فصل ششم

منطق فازی

Fuzzy Logic

shavandi@sharif.edu

Classical Logic

- **Proposition Logic**

Definition (Proposition) As in our ordinary informal language, "sentence" is used in the logic. Especially, a sentence having only "true (1)" or "false (0)" as its truth value is called "proposition".

Example 8.1 The following sentences are propositions.

Smith hits 30 home runs in one season.	(true)
$2 + 4 = 7$	(false)
For every x, if $f(x) = \sin x$, then $f'(x) = \cos x$.	(true)
It rains now.	(true) □

Classical Logic

Example 8.2. The followings are not propositions.

Why are you interested in the fuzzy theory?

He hits 5 home runs in one season.

$$x + 5 = 0$$

$$x + y = z$$

- In the second example, we do not know who is "He" and thus cannot determine whether the sentence is true (1) or false (0). If "He" is replaced by "Tom", we have

  Tom hits 5 home runs in one season.

  Now we can evaluate the truth value of the above sentence.

Classical Logic

- **Definition (Logic variable)** As we know now, a proposition has its value (true or false). If we represent a proposition as a variable, the variable can have the value true or false.
- This type of variable is called as a "proposition variable" or "logic variable".
- We can combine prepositional variables by using “connectives".
- The basic connectives: **negation, conjunction, disjunction, and implication.**

Classical Logic

Table 8.2. Truth table of conjunction

a	b	$a \wedge b$
1	1	1
1	0	0
0	1	0
0	0	0

Table 8.3. Truth table of disjunction

a	b	$a \vee b$
1	1	1
1	0	1
0	1	1
0	0	0

- **Implication**

  The proposition "if a, then b." is represented as follows.

$$a \to b$$

Classical Logic

Table 8.4. Truth table of implication

a	b	$a \to b$
1	1	1
1	0	0
0	1	1
0	0	1

i) $a \to b$ where a: $2 + 2 = 4$, b: $3 + 3 = 6$
ii) $a \to b$ where a: $2 + 2 = 4$, b: $3+3 = 7$
iii) $a \to b$ where a: $2 + 2 = 5$, b: $3 + 3 = 6$
iv) $a \to b$ where a: $2 + 2 = 5$, b: $3 + 3 = 7$

we can see that the above propositions are true except for the second.

Classical Logic

- **Logic Function**

  The "logic function" is a combination of propositional variables by using connectives. Values of the logic function can be evaluated according to the values of propositional variables and the truth values of connectives.

Definition (Logic formula) The logic formula is defined as following :

i) Truth values 0 and 1 are logic formulas

ii) If v is a logic variable, v and $\overline{v}$ are a logic formulas

iii) If a and b represent a logic formulas, a $\wedge$ b and a $\vee$ b are also logic formulas.

iv) The expressions defined by the above (1), (2), and (3) are logic formulas. □

Classical Logic

- Any logic formula defines a logic function, and it has its truth value.
- Properties of logic formulas are summarized in (Table 8.5.)
- Some of important logic formulas and their values are given in the following:

(1)	Negation	$\overline{a} = 1 - a$
(2)	Conjunction	$a \wedge b = \text{Min}\ (a, b)$
(3)	Disjunction	$a \vee b = \text{Max}\ (a, b)$
(4)	Implication	$a \rightarrow b = \overline{a} \vee b$

Table 8.5. Properties of classical logic

(1) Involution	$\overline{\overline{a}} = a$
(2) Commutativity	$a \wedge b = b \wedge a$
	$a \vee b = b \vee a$
(3) Associativity	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
	$(a \vee b) \vee c = a \vee (b \vee c)$
(4) Distributivity	$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
	$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge b)$
(5) Idempotency	$a \wedge a = a$
	$a \vee a = a$
(6) Absorption	$a \vee (a \wedge b) = a$
	$a \wedge (a \vee b) = a$
(7) Absorption by 0 and 1	$a \wedge 0 = 0$
	$a \vee 1 = 1$
(8) Identity	$a \wedge 1 = a$
	$a \vee 0 = a$
(9) De Morgan's law	$\overline{a \wedge b} = \overline{a} \vee \overline{b}$
	$\overline{a \vee b} = \overline{a} \wedge \overline{b}$
(10) Absorption of complement	$a \vee (\overline{a} \wedge b) = a \vee b$
	$a \wedge (\overline{a} \vee b) = a \wedge b$
(11) Law of contradiction	$a \wedge \overline{a} = 0$
(12) Law of excluded middle	$a \vee \overline{a} = 1$

Classical Logic

- **Tautology and Inference Rule**

Definition (Tautology) A "tautology" is a logic formula whose value is always true regardless of its logic variables. A "contradiction" is one which is always false. □

Example 8.5 Consider the following logic formula.

$$\overline{(a \to b)} \to \overline{b}$$

This proposition means that if the value of $(a \to b)$ is false then b is false. Let's evaluate its value with different values of logic variables a and b in (Table 8.6.)

Classical Logic

Tautology and Inference Rule

Table 8.6. Truth value of tautology $\overline{(a \rightarrow b)} \rightarrow \bar{b}$

a	b	$a \rightarrow b$	$\overline{(a \rightarrow b)}$	$\bar{b}$	$\overline{(a \rightarrow b)} \rightarrow \bar{b}$
1	1	1	0	0	1
1	0	0	1	1	1
0	1	1	0	0	1
0	0	1	0	1	1

in Table 8.6, we see that the logic formula is always true and thus it is a tautology. That is, if $\overline{(a \rightarrow b)}$ is true, then $\bar{b}$ is always true. □

Classical Logic

Tautology and Inference Rule

Example 8.6 Let's consider another example.

$$(a \wedge (a \rightarrow b)) \rightarrow b$$

The truth values of this proposition are evaluated in (Table 8.7.) we can see that this proposition has also true value regardless of the values of a and b. This tautology means that
"If *a* is true and ($a \rightarrow b$) is true, then *b* is true." or
"If *a* exists and the relation ($a \rightarrow b$) is true, then *b* exists."

Table 8.7. Truth value of tautology $(a \wedge (a \rightarrow b)) \rightarrow b$

a	b	$(a \rightarrow b)$	$(a \wedge (a \rightarrow b))$	$(a \wedge (a \rightarrow b)) \rightarrow b$
1	1	1	1	1
1	0	0	0	1
0	1	1	0	1
0	0	1	0	1

Classical Logic

- **Tautology and Inference Rule**
- The interpretations in the above example show a logic procedure of tautology, and thus we can obtain a correct conclusion when we follow the logic procedure.
- Therefore, the tautology is used as a rule of "deductive inference". There are some important inference rules using tautologies:

$(a \wedge (a \to b)) \to b$: modus ponens

$(\bar{b} \wedge (a \to b)) \to \bar{a}$: modus tollens

$((a \to b) \wedge (b \to c)) \to (a \to c)$: hypothetical syllogism

Fuzzy Sets. Chapter 6- Fuzzy Logic shavandi@sharif.edu

Classical Logic

- **Predicate Logic**

A “predicate” is a group of words like

“is a man”

“is green”

“is less than”

“belongs to”

They can be applied to one or more names of individuals (objects) to yield meaningful sentences; for example,

“Socrates is a man.”

“Two is less than four.”

“That hat belongs to me.”

“He is John.”

the names of the individuals are called individual constants.

Sharif University of Technology Industrial Engineering Dept.

Classical Logic

- **Predicate Logic**

Definition (Predicate logic) "Predicate logic" is a logic which represents a proposition with the predicate and individual (object). □

Example 8.7 The following propositions are "predicate propositions" and consist of predicates and objects.

"Socrates is a man"

predicate: "is a man"

object: "Socrates"

"Two is less than four"

predicate: "is less than"

object: "two" , "four"

Classical Logic

- **Quantifier**
- The phrase "for all" is called the "universal quantifier" and is denoted symbolically by $\forall$.
- The phrase "there exists", "there is a", or "for some" is called the "existential quantifier" and is denoted symbolically by $\exists$.

The universal quantifier is kind of an iterated conjunction. Suppose there are only finitely-many individuals. That is, the variable x takes only the values $a_1, a_2, \ldots a_n$. Then the sentence $\forall x P(x)$ has the same meaning as the conjunction $P(a_1) \wedge P(a_2) \wedge \ldots \wedge P(a_n)$.

The existential quantifier is kind of an iterated disjunction. If there are only finitely-many individuals $a_1, a_2, \ldots a_n$, then the sentence $\exists x P(x)$ has the same meaning as the disjunction $P(a_1) \vee P(a_2) \vee \ldots \vee P(a_n)$.

Fuzzy Logic

Fuzzy Expression

In the fuzzy expression(formula), a fuzzy proposition can have its truth value in the interval [0,1]. The fuzzy expression function is a mapping function from [0,1] to [0,1].

$$f : [0,1] \rightarrow [0,1]$$

If we generalize the domain in n-dimension, the function becomes as follows:

$$f : [0,1]^n \rightarrow [0,1]$$

Therefore we can interpret the fuzzy expression as an n-ary relation from n fuzzy sets to [0,1]. In the fuzzy logic, the operations such as negation (~ or ¬), conjunction (∧) and disjunction (∨) are used as in the classical logic.

Fuzzy Logic

Fuzzy Expression

Definition (Fuzzy logic) Then the fuzzy logic is a logic represented by the fuzzy expression (formula) which satisfies the followings.

i) Truth values, 0 and 1, and variable x_i ($\in [0,1]$, i = 1, 2, …, n) are fuzzy expressions.

ii) If f is a fuzzy expression, ~f is also a fuzzy expression.

iii) If f and g are fuzzy expressions, $f \wedge g$ and $f \vee g$ are also fuzzy expressions. □

Fuzzy Logic

- **Operators in Fuzzy Expression**

There are some operators in the fuzzy expression such as ¬ (negation), ∧ (conjunction), ∨ (disjunction), and → (implication). However the meaning of operators may be different according to the literature. If we follow Lukasiewicz's definition, the operators are defined as follows for a, b ∈ [0,1].

(1) Negation $\overline{a} = 1 - a$

(2) Conjunction $a \wedge b = \text{Min}\ (a, b)$

(3) Disjunction $a \vee b = \text{Max}\ (a, b)$

(4) Implication $a \rightarrow b = \text{Min}\ (1, 1+b-a)$

The properties of fuzzy operators are summarized in (Table 8.9.)

But we have to notice that the law of contradiction and law of excluded middle are not verified in the fuzzy logic.

Table 8.9. The properties of fuzzy logic operators

(1) Involution	$\overline{\overline{a}} = a$
(2) Commutativity	$a \wedge b = b \wedge a$
	$a \vee b = b \vee a$
(3) Associativity	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
	$(a \vee b) \vee c = a \vee (b \vee c)$
(4) Distributivity	$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
	$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(5) Idempotency	$a \wedge a = a$
	$a \vee a = a$
(6) Absorption	$a \vee (a \wedge b) = a$
	$a \wedge (a \vee b) = a$
(7) Absorption by 0 and 1	$a \wedge 0 = 0$
	$a \vee 1 = 1$
(8) Identity	$a \wedge 1 = a$
	$a \vee 0 = a$
(9) De Morgan's law	$\overline{a \wedge b} = \overline{a} \vee \overline{b}$
	$\overline{a \vee b} = \overline{a} \wedge \overline{b}$

Fuzzy Logic

- **Some Examples of Fuzzy Logic Operations**

  In this section, we have two examples of classical logic operation and one example of fuzzy logic operation. In these examples, we will see that the fuzzy logic operation is a generalization of the classical one

Example 8.12 When $a = 1, b = 0$

i) $\bar{a} = 0$
ii) $a \wedge b = \text{Min}(1, 0) = 0$
iii) $a \vee b = \text{Max}(1, 0) = 1$
iv) $a \to b = \text{Min}(1, 1-1+0) = 0$

Example 8.13 When $a = 1, b = 1$

i) $\bar{a} = 0$
ii) $a \wedge b = \text{Min}(1,1) = 1$
iii) $a \vee b = \text{Max}(1,1) = 1$
iv) $a \to b = \text{Min}(1, 1-1+1) = 1$

Example 8.14 When a = 0.6, b = 0.7

i) $\bar{a} = 0.4$
ii) $a \wedge b = \text{Min}(0.6,0.7) = 0.6$
iii) $a \vee b = \text{Max}(0.6, 0.7) = 0.7$
iv) $a \to b = \text{Min}(1, 1-0.6+0.7) = \text{Min}(1, 1.1) = 1$

Fuzzy Logic

- **Fuzzy Predicate**

Definition (Fuzzy predicate) .A fuzzy predicate is a predicate whose definition contains ambiguity □

Example 8.16 For example,

"z is expensive."
"w is young."

The terms "expensive" and "young" are fuzzy terms. Therefore the sets "expensive(z)" and "young(w)" are fuzzy sets. □

When a fuzzy predicate "x is P" is given, we can interpret it in two ways.

(1) P(x) is a fuzzy set. The membership degree of x in the set P is defined by the membership function $\mu_{P(x)}$.

(2) $\mu_{P(x)}$ is the satisfactory degree of x for the property P. Therefore, the truth value of the fuzzy predicate is defined by the membership function.

$$\text{Truth value} = \mu_{P(x)}$$

Fuzzy Logic

- **Fuzzy Modifier**

As we know, a new term can be obtained when we add the modifier "very" to a primary term. In this section we will see how semantic of the new term and membership function can be defined.

Example 8.17 Let's consider a linguistic variable "Age" in (Fig 8.1.) Linguistic terms "young" and "very young" are defined in the universal set U.

$$U = \{u \mid u \in [0,100]\}$$

The variable Age takes a value in the set T(Age).

$$T(Age) = \{young, very\ young, very\ very\ young, \ldots\}$$

In the figure, the term "young" is represented by a membership function $\mu_{young}(u)$. When we represent the term "very young", we can use the square of $\mu_{young}(u)$ as follows.

$$\mu_{very\ young}(u) = (\mu_{young}(u))^2$$

The graph of membership function of "very young" is given in the figure.□

Fuzzy Logic

- **Fuzzy Modifier**

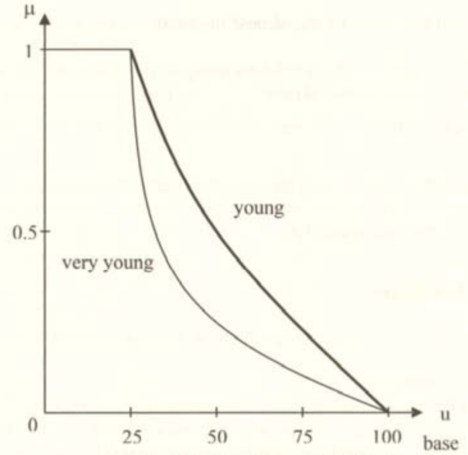

Fig. 8.1. Linguistic variable "Age"

Fuzzy Logic

- **Fuzzy Truth Qualifier**
- **Fuzzy Truth Values**

Baldwin defined fuzzy truth qualifier in the universal set $V = \{v \mid v \in [0,1]\}$ as follows.

T = {true, very true, fairly true, absolutely true, … , absolutely false, fairly false, false}

The qualifiers in T define "fuzzy truth values" and they can be defined by the membership functions. If we take baldwin's membership function $\mu_{true}(v)$, the truth qualifiers are represented by the followig membership functions(Fig 8.2).

Fuzzy Sets. Chapter 6- Fuzzy Logic
shavandi@sharif.edu

Fuzzy Logic

- **Fuzzy Truth Qualifier**
- **Fuzzy Truth Values**

$$\mu_{\text{true}}(v) = v \qquad v \in [0,1]$$
$$\mu_{\text{very true}}(v) = (\mu_{\text{true}}(v))^2 \qquad v \in [0,1]$$
$$\mu_{\text{fairly true}}(v) = (\mu_{\text{true}}(v))^{1/2} \qquad v \in [0,1]$$
$$\mu_{\text{false}}(v) = 1 - \mu_{\text{true}}(v) \qquad v \in [0,1]$$
$$\mu_{\text{very false}}(v) = (\mu_{\text{false}}(v))^2 \qquad v \in [0,1]$$
$$\mu_{\text{fairly false}}(v) = (\mu_{\text{false}}(v))^{1/2} \qquad v \in [0,1]$$
$$\mu_{\text{absolutely true}}(v) = \begin{cases} 1 & \text{for } v = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$\mu_{\text{absolutely false}}(v) = \begin{cases} 1 & \text{for } v = 0 \\ 0 & \text{otherwise} \end{cases}$$

shavandi@sharif.edu

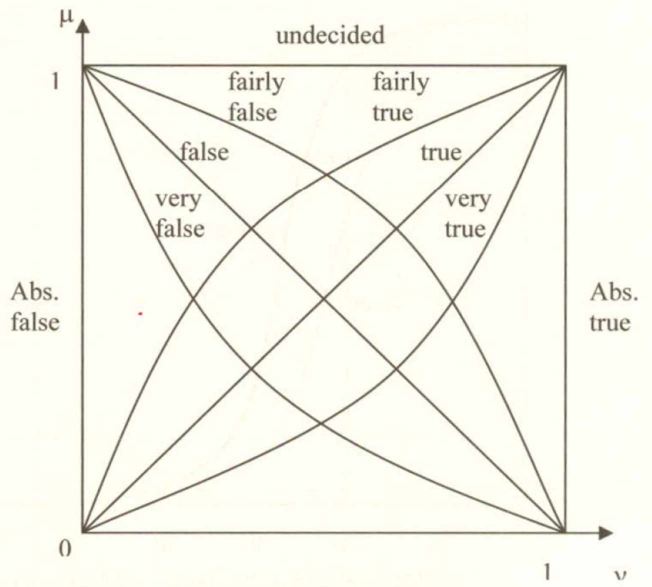

Fig. 8.2. Baldwin's truth graph

shavandi@sharif.edu

Representation of Fuzzy Rule

- **Inference and Knowledge Representation**

In general, the "inference" is a process to obtain new information by using existing knowledge. The representation of knowledge is an important issue in the inference. When we consider the representation methods, the following rule type "if-then" is the most popular form.

"If x is a, then y is b."

The rule is interpreted as an "implication" and consists of the "antecedent (if part)" and "consequent (then part)". If a rule is given in the above form and we have a fact in the following form,

"x is a"

then we can infer and obtain new result:

"y is b"

Representation of Fuzzy Rule

- **Inference and Knowledge Representation**

  Based on the above discussion, we can summarize two types of "reasoning" .

(1) Modus ponens
Fact: x is a
Rule: If x is a, then y is b
Result: y is b

(2) Modus tollens
Fact: y is $\bar{b}$
Rule: If x is a then y is b
Result: x is $\bar{a}$

The modus ponens is used in the forward inference and the modus tollens is in the backward one.

Representation of Fuzzy Rule

- **Representation of Fuzzy Predicate by Fuzzy Relation**

We saw that a fuzzy predicate is considered as a fuzzy set. In this section, we will see how the fuzzy predicate is used in fuzzy inference. When there is a fuzzy predicate proposition such that "x is P", it is represented by fuzzy set P(x) and whose membership function is by $\mu_{P(x)}(x)$. We know also a fuzzy relation is one type of fuzzy sets, and thus we can represent a predicate by using a relation.

"R(x) = P"

P is a fuzzy set and R(x) is a relation that consists of elements in P. The membership function of the predicate is represented by $\mu_{P(x)}(x)$ which shows the membership degree of x in P. The predicate represented by a relation will be used in the representation of fuzzy rule and premise.

Representation of Fuzzy Rule

Representation of Fuzzy Rule

When we consider fuzzy rules, the general form is given in the following.

If x is A, then y is B.

The fuzzy rule may include fuzzy predicates in the antecedent and consequent, and it can be rewritten as in the form.

If A(x), then B(y)

This rule can be represented by a relation R(x, y).

R(x, y): If A(x), then B(y)

or

$$R(x, y): A(x) \rightarrow B(y)$$

Representation of Fuzzy Rule

Representation of Fuzzy Rule

If there are a rule and facts involving fuzzy sets, we can execute two types of reasoning.

(1) Generalized modus ponens (GMP)

Fact:	x is A′	: R(x)
Rule:	If x is A then y is B	: R(x, y)
Result:	y is B′	: $R(y) = R(x) \circ R(x, y)$

(2) Generalized modus tollens (GMT)

Fact:	y is B′	: R(y)
Rule:	If x is A then y is B	: R(x, y)
Result:	x is A′	: $R(x) = R(y) \circ R(x, y)$

In the above reasoning, we see that the facts (A′ and B′) are not exactly same with the antecedents (A and B) in the rules; the results may be also different from the consequents. Therefore, we call this kind of inference as "fuzzy (approximate) reasoning or inference".

Representation of Fuzzy Rule

Representation of Fuzzy Rule

In general, when we execute the fuzzy (approximate) reasoning, we apply the "compositional rule of inference". The operation used in the reasoning is denoted by the notation "∘", and thus the result is represented by the output of the composition when we use the GMP.

$$R(y) = R(x) \circ R(x, y)$$

Representation of Fuzzy Rule

Representation of Fuzzy Rule

Example 8.20 We have knowledge such as :

If x is A then y is B

x is A′

From the above knowledge, how can we apply the inference procedure to get new information about y ?

i) We apply the implication operator to get implication relation $R(x,y) = A \times B$. Here, the cartesian product $A \times B$ is used.

$$R(x, y): A(x) \to B(y)$$

ii) We manipulate the fact into the form R(x) and then apply the generalized modus ponens

$$R(y) = R(x) \circ R(x, y)$$

In this step, composition operator "∘" is used. □

Therefore, there are two issues in the fuzzy reasoning: determination of the "implication relation" $R(x,y)$ and selection of the "composition operator". These issues will be discussed in the next chapter.