CHAPTER 6.

6.1. Atomic hydrogen contains 5.5×10^{25} atoms/m^3 at a certain temperature and pressure. When an electric field of 4 kV/m is applied, each dipole formed by the electron and positive nucleus has an effective length of 7.1×10^{-19} m.

a) Find P: With all identical dipoles, we have

$$P = Nqd = (5.5 \times 10^{25})(1.602 \times 10^{-19})(7.1 \times 10^{-19}) = 6.26 \times 10^{-12}\ \text{C/m}^2 = \underline{6.26\ \text{pC/m}^2}$$

b) Find ϵ_r: We use $P = \epsilon_0 \chi_e E$, and so

$$\chi_e = \frac{P}{\epsilon_0 E} = \frac{6.26 \times 10^{-12}}{(8.85 \times 10^{-12})(4 \times 10^3)} = 1.76 \times 10^{-4}$$

Then $\epsilon_r = 1 + \chi_e = \underline{1.000176}$.

6.2. Find the dielectric constant of a material in which the electric flux density is four times the polarization.

First we use $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + (1/4)\mathbf{D}$. Therefore $\mathbf{D} = (4/3)\epsilon_0 \mathbf{E}$, so we identify $\epsilon_r = \underline{4/3}$.

6.3. A coaxial conductor has radii $a = 0.8$ mm and $b = 3$ mm and a polystyrene dielectric for which $\epsilon_r = 2.56$. If $\mathbf{P} = (2/\rho)\mathbf{a}_\rho$ nC/m^2 in the dielectric, find:

a) $\mathbf{D}$ and $\mathbf{E}$ as functions of ρ: Use

$$\mathbf{E} = \frac{\mathbf{P}}{\epsilon_0(\epsilon_r - 1)} = \frac{(2/\rho) \times 10^{-9}\mathbf{a}_\rho}{(8.85 \times 10^{-12})(1.56)} = \underline{\frac{144.9}{\rho}\mathbf{a}_\rho\ \text{V/m}}$$

Then

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \frac{2 \times 10^{-9}\mathbf{a}_\rho}{\rho}\left[\frac{1}{1.56} + 1\right] = \frac{3.28 \times 10^{-9}\mathbf{a}_\rho}{\rho}\ \text{C/m}^2 = \underline{\frac{3.28\mathbf{a}_\rho}{\rho}\ \text{nC/m}^2}$$

b) Find V_{ab} and χ_e: Use

$$V_{ab} = -\int_3^{0.8} \frac{144.9}{\rho}\, d\rho = 144.9 \ln\left(\frac{3}{0.8}\right) = \underline{192\,\text{V}}$$

$\chi_e = \epsilon_r - 1 = \underline{1.56}$, as found in part a.

c) If there are 4×10^{19} molecules per cubic meter in the dielectric, find $\mathbf{p}(\rho)$: Use

$$\mathbf{p} = \frac{\mathbf{P}}{N} = \frac{(2 \times 10^{-9}/\rho)}{4 \times 10^{19}}\mathbf{a}_\rho = \underline{\frac{5.0 \times 10^{-29}}{\rho}\mathbf{a}_\rho\ \text{C} \cdot \text{m}}$$

6.4. Consider a composite material made up of two species, having number densities N_1 and N_2 molecules/m^3 respectively. The two materials are uniformly mixed, yielding a total number density of $N = N_1 + N_2$. The presence of an electric field $\mathbf{E}$, induces molecular dipole moments $\mathbf{p}_1$ and $\mathbf{p}_2$ within the individual species, whether mixed or not. Show that the dielectric constant of the composite material is given by $\epsilon_r = f\epsilon_{r1} + (1-f)\epsilon_{r2}$, where f is the number fraction of species 1 dipoles in the composite, and where ϵ_{r1} and ϵ_{r2} are the dielectric constants that the unmixed species would have if each had number density N.

We may write the total polarization vector as

$$\mathbf{P}_{tot} = N_1\mathbf{p}_1 + N_2\mathbf{p}_2 = N\left(\frac{N_1}{N}\mathbf{p}_1 + \frac{N_2}{N}\mathbf{p}_2\right) = N\left[f\mathbf{p}_1 + (1-f)\mathbf{p}_2\right] = f\mathbf{P}_1 + (1-f)\mathbf{P}_2$$

In terms of the susceptibilities, this becomes $\mathbf{P}_{tot} = \epsilon_0\left[f\chi_{e1} + (1-f)\chi_{e2}\right]\mathbf{E}$, where χ_{e1} and χ_{e2} are evaluated at the composite number density, N. Now

$$\mathbf{D} = \epsilon_r\epsilon_0\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}_{tot} = \epsilon_0\underbrace{\left[1 + f\chi_{e1} + (1-f)\chi_{e2}\right]}_{\epsilon_r}\mathbf{E}$$

Identifying ϵ_r as shown, we may rewrite it by adding and subracting f:

$$\begin{aligned}\epsilon_r &= \left[1 + f - f + f\chi_{e1} + (1-f)\chi_{e2}\right] = \left[f(1+\chi_{e1}) + (1-f)(1+\chi_{e2})\right] \\ &= \left[f\epsilon_{r1} + (1-f)\epsilon_{r2}\right] \quad \text{Q.E.D.}\end{aligned}$$

6.5. The surface $x = 0$ separates two perfect dielectrics. For $x > 0$, let $\epsilon_r = \epsilon_{r1} = 3$, while $\epsilon_{r2} = 5$ where $x < 0$. If $\mathbf{E}_1 = 80\mathbf{a}_x - 60\mathbf{a}_y - 30\mathbf{a}_z$ V/m, find:

a) E_{N1}: This will be $\mathbf{E}_1 \cdot \mathbf{a}_x = \underline{80\,\text{V/m}}$.

b) $\mathbf{E}_{T1}$. This has components of $\mathbf{E}_1$ *not* normal to the surface, or $\mathbf{E}_{T1} = \underline{-60\mathbf{a}_y - 30\mathbf{a}_z\ \text{V/m}}$.

c) $E_{T1} = \sqrt{(60)^2 + (30)^2} = \underline{67.1\,\text{V/m}}$.

d) $E_1 = \sqrt{(80)^2 + (60)^2 + (30)^2} = \underline{104.4\,\text{V/m}}$.

e) The angle θ_1 between $\mathbf{E}_1$ and a normal to the surface: Use

$$\cos\theta_1 = \frac{\mathbf{E}_1 \cdot \mathbf{a}_x}{E_1} = \frac{80}{104.4} \quad\Rightarrow\quad \theta_1 = \underline{40.0^\circ}$$

f) $D_{N2} = D_{N1} = \epsilon_{r1}\epsilon_0 E_{N1} = 3(8.85 \times 10^{-12})(80) = \underline{2.12\,\text{nC/m}^2}$.

g) $D_{T2} = \epsilon_{r2}\epsilon_0 E_{T1} = 5(8.85 \times 10^{-12})(67.1) = \underline{2.97\,\text{nC/m}^2}$.

h) $\mathbf{D}_2 = \epsilon_{r1}\epsilon_0 E_{N1}\mathbf{a}_x + \epsilon_{r2}\epsilon_0\mathbf{E}_{T1} = \underline{2.12\mathbf{a}_x - 2.66\mathbf{a}_y - 1.33\mathbf{a}_z\ \text{nC/m}^2}$.

i) $\mathbf{P}_2 = \mathbf{D}_2 - \epsilon_0\mathbf{E}_2 = \mathbf{D}_2\left[1 - (1/\epsilon_{r2})\right] = (4/5)\mathbf{D}_2 = \underline{1.70\mathbf{a}_x - 2.13\mathbf{a}_y - 1.06\mathbf{a}_z\ \text{nC/m}^2}$.

j) the angle θ_2 between $\mathbf{E}_2$ and a normal to the surface: Use

$$\cos\theta_2 = \frac{\mathbf{E}_2 \cdot \mathbf{a}_x}{E_2} = \frac{\mathbf{D}_2 \cdot \mathbf{a}_x}{D_2} = \frac{2.12}{\sqrt{(2.12)^2 = (2.66)^2 + (1.33)^2}} = .581$$

Thus $\theta_2 = \cos^{-1}(.581) = \underline{54.5^\circ}$.

6.6. The potential field in a slab of dielectric material for which $\epsilon_r = 1.6$ is given by $V = -5000x$.

a) Find $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$ in the material.

First, $\mathbf{E} = -\nabla V = 5000\,\mathbf{a}_x$ V/m. Then $\mathbf{D} = \epsilon_r\epsilon_0\mathbf{E} = 1.6\epsilon_0(5000)\,\mathbf{a}_x = \underline{70.8\,\mathbf{a}_x \text{ nC/m}^2}$. Then, $\chi_e = \epsilon_r - 1 = 0.6$, and so $\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = 0.6\epsilon_0(5000)\,\mathbf{a}_x = \underline{26.6\,\mathbf{a}_x \text{ nC/m}^2}$.

b) Evaluate ρ_v, ρ_b, and ρ_t in the material. Using the results in part a, we find $\rho_v = \nabla\cdot\mathbf{D} = \underline{0}$, $\rho_b = -\nabla\cdot\mathbf{P} = \underline{0}$, and $\rho_t = \nabla\cdot\epsilon_0\mathbf{E} = \underline{0}$.

6.7. Two perfect dielectrics have relative permittivities $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 8$. The planar interface between them is the surface $x - y + 2z = 5$. The origin lies in region 1. If $\mathbf{E}_1 = 100\mathbf{a}_x + 200\mathbf{a}_y - 50\mathbf{a}_z$ V/m, find $\mathbf{E}_2$: We need to find the components of $\mathbf{E}_1$ that are normal and tangent to the boundary, and then apply the appropriate boundary conditions. The normal component will be $E_{N1} = \mathbf{E}_1 \cdot \mathbf{n}$. Taking $f = x - y + 2z$, the unit vector that is normal to the surface is

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}}\left[\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z\right]$$

This normal will point in the direction of increasing f, which will be away from the origin, or into region 2 (you can visualize a portion of the surface as a triangle whose vertices are on the three coordinate axes at $x = 5$, $y = -5$, and $z = 2.5$). So $E_{N1} = (1/\sqrt{6})[100 - 200 - 100] = -81.7$ V/m. Since the magnitude is negative, the normal component points into region 1 from the surface. Then

$$\mathbf{E}_{N1} = -81.65\left(\frac{1}{\sqrt{6}}\right)\left[\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z\right] = -33.33\mathbf{a}_x + 33.33\mathbf{a}_y - 66.67\mathbf{a}_z \text{ V/m}$$

Now, the tangential component will be $\mathbf{E}_{T1} = \mathbf{E}_1 - \mathbf{E}_{N1} = 133.3\mathbf{a}_x + 166.7\mathbf{a}_y + 16.67\mathbf{a}_z$. Our boundary conditions state that $\mathbf{E}_{T2} = \mathbf{E}_{T1}$ and $\mathbf{E}_{N2} = (\epsilon_{r1}/\epsilon_{r2})\mathbf{E}_{N1} = (1/4)\mathbf{E}_{N1}$. Thus

$$\mathbf{E}_2 = \mathbf{E}_{T2} + \mathbf{E}_{N2} = \mathbf{E}_{T1} + \frac{1}{4}\mathbf{E}_{N1} = 133.3\mathbf{a}_x + 166.7\mathbf{a}_y + 16.67\mathbf{a}_z - 8.3\mathbf{a}_x + 8.3\mathbf{a}_y - 16.67\mathbf{a}_z$$
$$= \underline{125\mathbf{a}_x + 175\mathbf{a}_y \text{ V/m}}$$

6.8. Region 1 ($x \geq 0$) is a dielectric with $\epsilon_{r1} = 2$, while region 2 ($x < 0$) has $\epsilon_{r2} = 5$. Let $\mathbf{E}_1 = 20\mathbf{a}_x - 10\mathbf{a}_y + 50\mathbf{a}_z$ V/m.

a) Find $\mathbf{D}_2$: One approach is to first find $\mathbf{E}_2$. This will have the same y and z (tangential) components as $\mathbf{E}_1$, but the normal component, E_x, will differ by the ratio $\epsilon_{r1}/\epsilon_{r2}$; this arises from $D_{x1} = D_{x2}$ (normal component of $\mathbf{D}$ is continuous across a non-charged interface). Therefore $\mathbf{E}_2 = 20(\epsilon_{r1}/\epsilon_{r2})\,\mathbf{a}_x - 10\,\mathbf{a}_y + 50\,\mathbf{a}_z = 8\,\mathbf{a}_x - 10\,\mathbf{a}_y + 50\,\mathbf{a}_z$. The flux density is then

$$\mathbf{D}_2 = \epsilon_{r2}\epsilon_0\mathbf{E}_2 = 40\epsilon_0\,\mathbf{a}_x - 50\epsilon_0\,\mathbf{a}_y + 250\epsilon_0\,\mathbf{a}_z = \underline{0.35\,\mathbf{a}_x - 0.44\,\mathbf{a}_y + 2.21\,\mathbf{a}_z \text{ nC/m}^2}$$

b) Find the energy density in both regions: These will be

$$w_{e1} = \frac{1}{2}\epsilon_{r1}\epsilon_0\mathbf{E}_1\cdot\mathbf{E}_1 = \frac{1}{2}(2)\epsilon_0\left[(20)^2 + (10)^2 + (50)^2\right] = 3000\epsilon_0 = \underline{26.6 \text{ nJ/m}^3}$$

$$w_{e2} = \frac{1}{2}\epsilon_{r2}\epsilon_0\mathbf{E}_2\cdot\mathbf{E}_2 = \frac{1}{2}(5)\epsilon_0\left[(8)^2 + (10)^2 + (50)^2\right] = 6660\epsilon_0 = \underline{59.0 \text{ nJ/m}^3}$$

6.9. Let the cylindrical surfaces $\rho = 4$ cm and $\rho = 9$ cm enclose two wedges of perfect dielectrics, $\epsilon_{r1} = 2$ for $0 < \phi < \pi/2$, and $\epsilon_{r2} = 5$ for $\pi/2 < \phi < 2\pi$. If $\mathbf{E}_1 = (2000/\rho)\mathbf{a}_\rho$ V/m, find:

a) $\mathbf{E}_2$: The interfaces between the two media will lie on planes of constant ϕ, to which $\mathbf{E}_1$ is parallel. Thus the field is the same on either side of the boundaries, and so $\underline{\mathbf{E}_2 = \mathbf{E}_1}$.

b) the total electrostatic energy stored in a 1m length of each region: In general we have $w_E = (1/2)\epsilon_r\epsilon_0 E^2$. So in region 1:

$$W_{E1} = \int_0^1 \int_0^{\pi/2} \int_4^9 \frac{1}{2}(2)\epsilon_0 \frac{(2000)^2}{\rho_2} \rho\, d\rho\, d\phi\, dz = \frac{\pi}{2}\epsilon_0 (2000)^2 \ln\left(\frac{9}{4}\right) = \underline{45.1\,\mu\text{J}}$$

In region 2, we have

$$W_{E2} = \int_0^1 \int_{\pi/2}^{2\pi} \int_4^9 \frac{1}{2}(5)\epsilon_0 \frac{(2000)^2}{\rho_2} \rho\, d\rho\, d\phi\, dz = \frac{15\pi}{4}\epsilon_0 (2000)^2 \ln\left(\frac{9}{4}\right) = \underline{338\,\mu\text{J}}$$

6.10. Let $S = 100\,\text{mm}^2$, $d = 3$ mm, and $\epsilon_r = 12$ for a parallel-plate capacitor.

a) Calculate the capacitance:

$$C = \frac{\epsilon_r\epsilon_0 A}{d} = \frac{12\epsilon_0(100 \times 10^{-6})}{3 \times 10^{-3}} = 0.4\epsilon_0 = \underline{3.54 \text{ pf}}$$

b) After connecting a 6 V battery across the capacitor, calculate E, D, Q, and the total stored electrostatic energy: First,

$$E = V_0/d = 6/(3 \times 10^{-3}) = \underline{2000 \text{ V/m}}, \quad \text{then} \quad D = \epsilon_r\epsilon_0 E = 2.4 \times 10^4 \epsilon_0 = \underline{0.21\ \mu\text{C/m}^2}$$

The charge in this case is

$$Q = \mathbf{D} \cdot \mathbf{n}|_s = DA = 0.21 \times (100 \times 10^{-6}) = 0.21 \times 10^{-4}\ \mu\text{C} = \underline{21 \text{ pC}}$$

Finally, $W_e = (1/2)QV_0 = 0.5(21)(6) = \underline{63 \text{ pJ}}$.

c) With the source still connected, the dielectric is carefully withdrawn from between the plates. With the dielectric gone, re-calculate E, D, Q, and the energy stored in the capacitor.

$$E = V_0/d = 6/(3 \times 10^{-3}) = \underline{2000 \text{ V/m}}, \quad \text{as before.} \quad D = \epsilon_0 E = 2000\epsilon_0 = \underline{17.7 \text{ nC/m}^2}$$

The charge is now $Q = DA = 17.7 \times (100 \times 10^{-6})$ nC $= \underline{1.8 \text{ pC}}$.
Finally, $W_e = (1/2)QV_0 = 0.5(1.8)(6) = \underline{5.4 \text{ pJ}}$.

d) If the charge and energy found in (c) are less than that found in (b) (which you should have discovered), what became of the missing charge and energy? In the absence of friction in removing the dielectric, the charge and energy have returned to the battery that gave it.

6.11. Capacitors tend to be more expensive as their capacitance and maximum voltage, V_{max}, increase. The voltage V_{max} is limited by the field strength at which the dielectric breaks down, E_{BD}. Which of these dielectrics will give the largest CV_{max} product for equal plate areas: (a) air: $\epsilon_r = 1$, $E_{BD} = 3$ MV/m; (b) barium titanate: $\epsilon_r = 1200$, $E_{BD} = 3$ MV/m; (c) silicon dioxide: $\epsilon_r = 3.78$, $E_{BD} = 16$ MV/m; (d) polyethylene: $\epsilon_r = 2.26$, $E_{BD} = 4.7$ MV/m? Note that $V_{max} = EBDd$, where d is the plate separation. Also, $C = \epsilon_r \epsilon_0 A/d$, and so $V_{max}C = \epsilon_r \epsilon_0 A E_{BD}$, where A is the plate area. The maximum CV_{max} product is found through the maximum $\epsilon_r E_{BD}$ product. Trying this with the given materials yields the winner, which is barium titanate.

6.12. An air-filled parallel-plate capacitor with plate separation d and plate area A is connected to a battery which applies a voltage V_0 between plates. With the battery left connected, the plates are moved apart to a distance of $10d$. Determine by what factor each of the following quantities changes:

a) V_0: Remains the same, since the battery is left connected.

b) C: As $C = \epsilon_0 A/d$, increasing d by a factor of ten decreases C by a factor of 0.1.

c) E: We require $E \times d = V_0$, where V_0 has not changed. Therefore, E has decreased by a factor of 0.1.

d) D: As $D = \epsilon_0 E$, and since E has decreased by 0.1, D decreases by 0.1.

e) Q: Since $Q = CV_0$, and as C is down by 0.1, Q also decreases by 0.1.

f) ρ_S: As Q is reduced by 0.1, ρ_S reduces by 0.1. This is also consistent with D having been reduced by 0.1.

g) W_e: Use $W_e = 1/2\, CV_0^2$, to observe its reduction by 0.1, since C is reduced by that factor.

6.13. A parallel plate capacitor is filled with a nonuniform dielectric characterized by $\epsilon_r = 2 + 2 \times 10^6 x^2$, where x is the distance from one plate. If $S = 0.02\,\text{m}^2$, and $d = 1$ mm, find C: Start by assuming charge density ρ_s on the top plate. **D** will, as usual, be x-directed, originating at the top plate and terminating on the bottom plate. The key here is that **D** *will be constant over the distance between plates*. This can be understood by considering the x-varying dielectric as constructed of many thin layers, each having constant permittivity. The permittivity changes from layer to layer to approximate the given function of x. The approximation becomes exact as the layer thicknesses approach zero. We know that **D**, which is normal to the layers, will be continuous across each boundary, and so **D** is constant over the plate separation distance, and will be given in magnitude by ρ_s. The electric field magnitude is now

$$E = \frac{D}{\epsilon_0 \epsilon_r} = \frac{\rho_s}{\epsilon_0 (2 + 2 \times 10^6 x^2)}$$

The voltage beween plates is then

$$V_0 = \int_0^{10^{-3}} \frac{\rho_s\, dx}{\epsilon_0 (2 + 2 \times 10^6 x^2)} = \frac{\rho_s}{\epsilon_0} \frac{1}{\sqrt{4 \times 10^6}} \tan^{-1}\left(\frac{x\sqrt{4 \times 10^6}}{2} \right) \Big|_0^{10^{-3}} = \frac{\rho_s}{\epsilon_0} \frac{1}{2 \times 10^3} \left(\frac{\pi}{4} \right)$$

Now $Q = \rho_s(.02)$, and so

$$C = \frac{Q}{V_0} = \frac{\rho_s (.02) \epsilon_0 (2 \times 10^3)(4)}{\rho_s \pi} = 4.51 \times 10^{-10}\,\text{F} = \underline{451\,\text{pF}}$$

6.14. Repeat Problem 6.12 assuming the battery is disconnected before the plate separation is increased: The ordering of parameters is changed over that in Problem 6.12, as the progression of thought on the matter is different.

a) Q: Remains the <u>same</u>, since with the battery disconnected, the charge has nowhere to go.

b) ρ_S: As Q is unchanged, ρ_S is also <u>unchanged</u>, since the plate area is the same.

c) D: As $D = \rho_S$, it will remain the <u>same</u> also.

d) E: Since $E = D/\epsilon_0$, and as D is not changed, E will also remain the <u>same</u>.

e) V_0: We require $E \times d = V_0$, where E has not changed. Therefore, V_0 has increased by a factor of <u>10</u>.

f) C: As $C = \epsilon_0 A/d$, increasing d by a factor of ten decreases C by a factor of <u>0.1</u>. The same result occurs because $C = Q/V_0$, where V_0 is increased by 10, whereas Q has not changed.

g) W_e: Use $W_e = 1/2\, C V_0^2 = 1/2\, Q V_0$, to observe its increase by a factor of <u>10</u>.

6.15. Let $\epsilon_{r1} = 2.5$ for $0 < y < 1$ mm, $\epsilon_{r2} = 4$ for $1 < y < 3$ mm, and ϵ_{r3} for $3 < y < 5$ mm. Conducting surfaces are present at $y = 0$ and $y = 5$ mm. Calculate the capacitance per square meter of surface area if: a) ϵ_{r3} is that of air; b) $\epsilon_{r3} = \epsilon_{r1}$; c) $\epsilon_{r3} = \epsilon_{r2}$; d) region 3 is silver: The combination will be three capacitors in series, for which

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{d_1}{\epsilon_{r1}\epsilon_0(1)} + \frac{d_2}{\epsilon_{r2}\epsilon_0(1)} + \frac{d_3}{\epsilon_{r3}\epsilon_0(1)} = \frac{10^{-3}}{\epsilon_0}\left[\frac{1}{2.5} + \frac{2}{4} + \frac{2}{\epsilon_{r3}}\right]$$

So that

$$C = \frac{(5 \times 10^{-3})\epsilon_0 \epsilon_{r3}}{10 + 4.5\epsilon_{r3}}$$

Evaluating this for the four cases, we find a) $C = \underline{3.05\,\text{nF}}$ for $\epsilon_{r3} = 1$, b) $C = \underline{5.21\,\text{nF}}$ for $\epsilon_{r3} = 2.5$, c) $C = \underline{6.32\,\text{nF}}$ for $\epsilon_{r3} = 4$, and d) $C = \underline{9.83\,\text{nF}}$ if silver (taken as a perfect conductor) forms region 3; this has the effect of removing the term involving ϵ_{r3} from the original formula (first equation line), or equivalently, allowing ϵ_{r3} to approach infinity.

6.16. A parallel-plate capacitor is made using two circular plates of radius a, with the bottom plate on the xy plane, centered at the origin. The top plate is located at $z = d$, with its center on the z axis. Potential V_0 is on the top plate; the bottom plate is grounded. Dielectric having *radially-dependent* permittivity fills the region between plates. The permittivity is given by $\epsilon(\rho) = \epsilon_0(1 + \rho/a)$. Find:

a) **E**: Since ϵ does not vary in the z direction, and since we must always obtain V_0 when integrating **E** between plates, it must follow that $\mathbf{E} = \underline{-V_0/d\, \mathbf{a}_z \text{ V/m}}$.

b) **D**: $\mathbf{D} = \epsilon \mathbf{E} = \underline{-[\epsilon_0(1 + \rho/a)V_0/d]\; \mathbf{a}_z \text{ C/m}^2}$.

c) Q: Here we find the integral of the surface charge density over the top plate:

$$Q = \int_S \mathbf{D} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^a \frac{-\epsilon_0(1 + \rho/a)V_0}{d}\, \mathbf{a}_z \cdot (-\mathbf{a}_z)\, \rho\, d\rho\, d\phi = \frac{2\pi\epsilon_0 V_0}{d}\int_0^a (\rho + \rho^2/a)\, d\rho$$
$$= \frac{5\pi\epsilon_0 a^2}{3d} V_0$$

d) C: We use $C = Q/V_0$ and our previous result to find $C = \underline{5\epsilon_0(\pi a^2)/(3d) \text{ F}}$.

6.17. Two coaxial conducting cylinders of radius 2 cm and 4 cm have a length of 1m. The region between the cylinders contains a layer of dielectric from $\rho = c$ to $\rho = d$ with $\epsilon_r = 4$. Find the capacitance if

a) $c = 2$ cm, $d = 3$ cm: This is two capacitors in series, and so

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2\pi\epsilon_0}\left[\frac{1}{4}\ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right)\right] \Rightarrow C = \underline{143\,\text{pF}}$$

b) $d = 4$ cm, and the volume of the dielectric is the same as in part a: Having equal volumes requires that $3^2 - 2^2 = 4^2 - c^2$, from which $c = 3.32$ cm. Now

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2\pi\epsilon_0}\left[\ln\left(\frac{3.32}{2}\right) + \frac{1}{4}\ln\left(\frac{4}{3.32}\right)\right] \Rightarrow C = \underline{101\,\text{pF}}$$

6.18. (a) If we could specify a material to be used as the dielectric in a coaxial capacitor for which the permittivity varied continuously with radius, what variation with ρ should be used in order to maintain a uniform value of the electric field intensity?

Gauss's law tells us that regardless of the radially-varying permittivity, $\mathbf{D} = (a\rho_s/\rho)\,\mathbf{a}_\rho$, where a is the inner radius and ρ_s is the presumed surface charge density on the inner cylinder. Now

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{a\rho_s}{\epsilon\rho}\,\mathbf{a}_\rho$$

which indicates that ϵ must have a $\underline{1/\rho}$ dependence if $\mathbf{E}$ is to be constant with radius.

b) Under the conditions of part a, how do the inner and outer radii appear in the expression for the capacitance per unit distance? Let $\epsilon = g/\rho$ where g is a constant. Then $\mathbf{E} = a\rho_s/g\,\mathbf{a}_\rho$ and the voltage between cylinders will be

$$V_0 = -\int_b^a \frac{a\rho_s}{g}\,\mathbf{a}_\rho \cdot \mathbf{a}_\rho\,d\rho = \frac{a\rho_s}{g}(b-a)$$

where b is the outer radius. The capacitance per unit length is then $C = 2\pi a\rho_s/V_0 = 2\pi g/(b-a)$, or a simple inverse-distance relation.

6.19. Two conducting spherical shells have radii $a = 3$ cm and $b = 6$ cm. The interior is a perfect dielectric for which $\epsilon_r = 8$.

a) Find C: For a spherical capacitor, we know that:

$$C = \frac{4\pi\epsilon_r\epsilon_0}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi(8)\epsilon_0}{\left(\frac{1}{3} - \frac{1}{6}\right)(100)} = 1.92\pi\epsilon_0 = \underline{53.3\,\text{pF}}$$

b) A portion of the dielectric is now removed so that $\epsilon_r = 1.0$, $0 < \phi < \pi/2$, and $\epsilon_r = 8$, $\pi/2 < \phi < 2\pi$. Again, find C: We recognize here that removing that portion leaves us with two capacitors in parallel (whose C's will add). We use the fact that with the dielectric *completely* removed, the capacitance would be $C(\epsilon_r = 1) = 53.3/8 = 6.67\,\text{pF}$. With one-fourth the dielectric removed, the total capacitance will be

$$C = \frac{1}{4}(6.67) + \frac{3}{4}(53.4) = \underline{41.7\,\text{pF}}$$

6.20. Show that the capacitance per unit length of a cylinder of radius a is zero: Let ρ_s be the surface charge density on the surface at $\rho = a$. Then the charge per unit length is $Q = 2\pi a \rho_s$. The electric field (assuming free space) is $\mathbf{E} = (a\rho_s)/(\epsilon_0 \rho)\,\mathbf{a}_\rho$. The potential difference is evaluated between radius a and infinite radius, and is

$$V_0 = -\int_{\infty}^{a} \frac{a\rho_s}{\epsilon_0 \rho}\,\mathbf{a}_\rho \cdot \mathbf{a}_\rho\, d\rho \;\rightarrow\; \infty$$

The capacitance, equal to Q/V_0, is therefore zero.

6.21. With reference to Fig. 6.9, let $b = 6$ m, $h = 15$ m, and the conductor potential be 250 V. Take $\epsilon = \epsilon_0$. Find values for K_1, ρ_L, a, and C: We have

$$K_1 = \left[\frac{h + \sqrt{h^2 + b^2}}{b}\right]^2 = \left[\frac{15 + \sqrt{(15)^2 + (6)^2}}{6}\right]^2 = \underline{23.0}$$

We then have

$$\rho_L = \frac{4\pi\epsilon_0 V_0}{\ln K_1} = \frac{4\pi\epsilon_0(250)}{\ln(23)} = \underline{8.87\,\text{nC/m}}$$

Next, $a = \sqrt{h^2 - b^2} = \sqrt{(15)^2 - (6)^2} = \underline{13.8\,\text{m}}$. Finally,

$$C = \frac{2\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{2\pi\epsilon_0}{\cosh^{-1}(15/6)} = \underline{35.5\,\text{pF}}$$

6.22. Two #16 copper conductors (1.29-mm diameter) are parallel with a separation d between axes. Determine d so that the capacitance between wires in air is 30 pF/m.

We use

$$\frac{C}{L} = 60 \text{ pF/m} = \frac{2\pi\epsilon_0}{\cosh^{-1}(h/b)}$$

The above expression evaluates the capacitance of one of the wires suspended over a plane at mid-span, $h = d/2$. Therefore the capacitance of that structure is doubled over that required (from 30 to 60 pF/m). Using this,

$$\frac{h}{b} = \cosh\left(\frac{2\pi\epsilon_0}{C/L}\right) = \cosh\left(\frac{2\pi \times 8.854}{60}\right) = 1.46$$

Therefore, $d = 2h = 2b(1.46) = 2(1.29/2)(1.46) = \underline{1.88 \text{ mm}}$.

6.23. A 2 cm diameter conductor is suspended in air with its axis 5 cm from a conducting plane. Let the potential of the cylinder be 100 V and that of the plane be 0 V. Find the surface charge density on the:

a) cylinder at a point nearest the plane: The cylinder will image across the plane, producing an equivalent two-cylinder problem, with the second one at location 5 cm below the plane. We will take the plane as the zy plane, with the cylinder positions at $x = \pm 5$. Now $b = 1$ cm, $h = 5$ cm, and $V_0 = 100$ V. Thus $a = \sqrt{h^2 - b^2} = 4.90$ cm. Then $K_1 = [(h + a)/b]^2 = 98.0$, and $\rho_L = (4\pi\epsilon_0 V_0)/\ln K_1 = 2.43\,\text{nC/m}$. Now

$$\mathbf{D} = \epsilon_0 \mathbf{E} = -\frac{\rho_L}{2\pi}\left[\frac{(x + a)\mathbf{a}_x + y\mathbf{a}_y}{(x + a)^2 + y^2} - \frac{(x - a)\mathbf{a}_x + y\mathbf{a}_y}{(x - a)^2 + y^2}\right]$$

6.23a. (continued)

and

$$\rho_{s,\,max} = \mathbf{D}\cdot(-\mathbf{a}_x)\Big|_{x=h-b,y=0} = \frac{\rho_L}{2\pi}\left[\frac{h-b+a}{(h-b+a)^2} - \frac{h-b-a}{(h-b-a)^2}\right] = \underline{473\,\mathrm{nC/m^2}}$$

b) plane at a point nearest the cylinder: At $x = y = 0$,

$$\mathbf{D}(0,0) = -\frac{\rho_L}{2\pi}\left[\frac{a\mathbf{a}_x}{a^2} - \frac{-a\mathbf{a}_x}{a^2}\right] = -\frac{\rho_L}{2\pi}\frac{2}{a}\mathbf{a}_x$$

from which

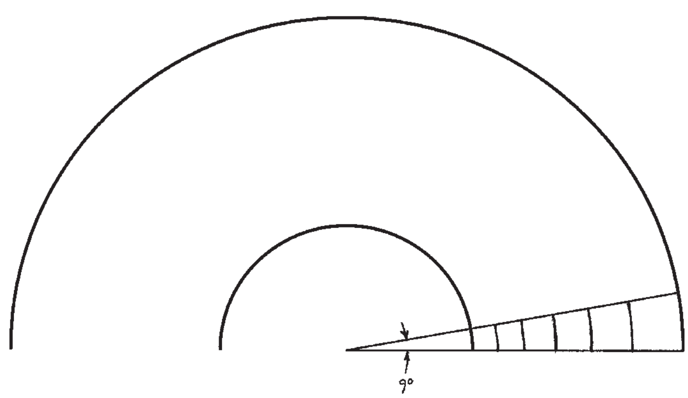
$$\rho_s = \mathbf{D}(0,0)\cdot\mathbf{a}_x = -\frac{\rho_L}{\pi a} = \underline{-15.8\,\mathrm{nC/m^2}}$$

6.24. For the conductor configuration of Problem 6.23, determine the capacitance per unit length. This is a quick one if we have already solved 6.23. The capacitance per unit length will be $C = \rho_L/V_0 = 2.43$ [nC/m]/100 = $\underline{24.3\text{ pF/m}}$.

6.25 Construct a curvilinear square map for a coaxial capacitor of 3-cm inner radius and 8-cm outer radius. These dimensions are suitable for the drawing.

a) Use your sketch to calculate the capacitance per meter length, assuming $\epsilon_R = 1$: The sketch is shown below. Note that only a 9° sector was drawn, since this would then be duplicated 40 times around the circumference to complete the drawing. The capacitance is thus

$$C \doteq \epsilon_0 \frac{N_Q}{N_V} = \epsilon_0 \frac{40}{6} = \underline{59\text{ pF/m}}$$



b) Calculate an exact value for the capacitance per unit length: This will be

$$C = \frac{2\pi\epsilon_0}{\ln(8/3)} = \underline{57\text{ pF/m}}$$

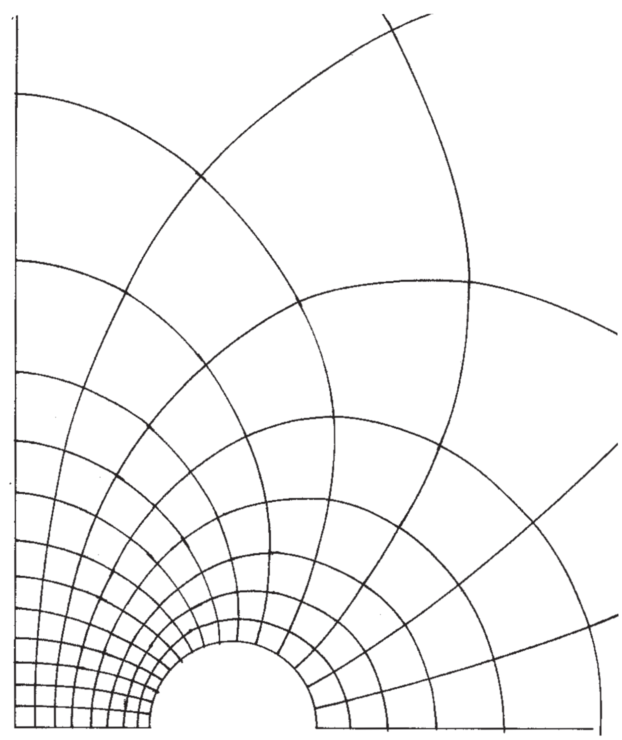
6.26 Construct a curvilinear-square map of the potential field about two parallel circular cylinders, each of 2.5 cm radius, separated by a center-to-center distance of 13cm. These dimensions are suitable for the actual sketch if symmetry is considered. As a check, compute the capacitance per meter both from your sketch and from the exact formula. Assume $\epsilon_R = 1$.

Symmetry allows us to plot the field lines and equipotentials over just the first quadrant, as is done in the sketch below (shown to one-half scale). The capacitance is found from the formula $C = (N_Q/N_V)\epsilon_0$, where N_Q is twice the number of squares around the perimeter of the half-circle and N_V is twice the number of squares between the half-circle and the left vertical plane. The result is

$$C = \frac{N_Q}{N_V}\epsilon_0 = \frac{32}{16}\epsilon_0 = 2\epsilon_0 = \underline{17.7 \text{ pF/m}}$$

We check this result with that using the exact formula:

$$C = \frac{\pi\epsilon_0}{\cosh^{-1}(d/2a)} = \frac{\pi\epsilon_0}{\cosh^{-1}(13/5)} = 1.95\epsilon_0 = \underline{17.3 \text{ pF/m}}$$



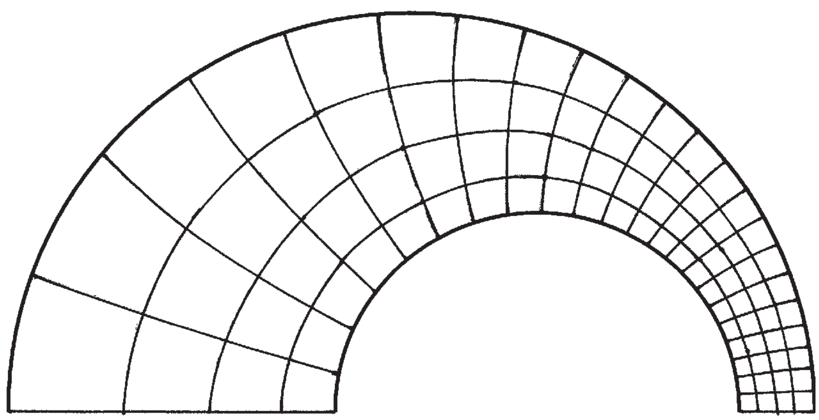
6.27. Construct a curvilinear square map of the potential field between two parallel circular cylinders, one of 4-cm radius inside one of 8-cm radius. The two axes are displaced by 2.5 cm. These dimensions are suitable for the drawing. As a check on the accuracy, compute the capacitance per meter from the sketch and from the exact expression:

$$C = \frac{2\pi\epsilon}{\cosh^{-1}\left[(a^2 + b^2 - D^2)/(2ab)\right]}$$

where a and b are the conductor radii and D is the axis separation.

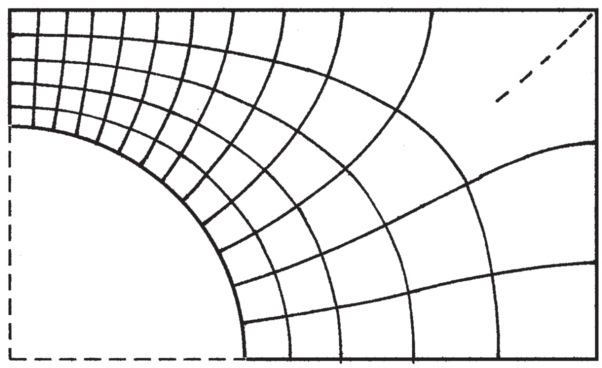
The drawing is shown below. Use of the exact expression above yields a capacitance value of $C = \underline{11.5\epsilon_0 \text{ F/m}}$. Use of the drawing produces:

$$C \doteq \frac{22 \times 2}{4}\epsilon_0 = \underline{11\epsilon_0 \text{ F/m}}$$



6.28. A solid conducting cylinder of 4-cm radius is centered within a rectangular conducting cylinder with a 12-cm by 20-cm cross-section.

a) Make a full-size sketch of one quadrant of this configuration and construct a curvilinear-square map for its interior: The result below could still be improved a little, but is nevertheless sufficient for a reasonable capacitance estimate. Note that the five-sided region in the upper right corner has been partially subdivided (dashed line) in anticipation of how it would look when the next-level subdivision is done (doubling the number of field lines and equipotentials).

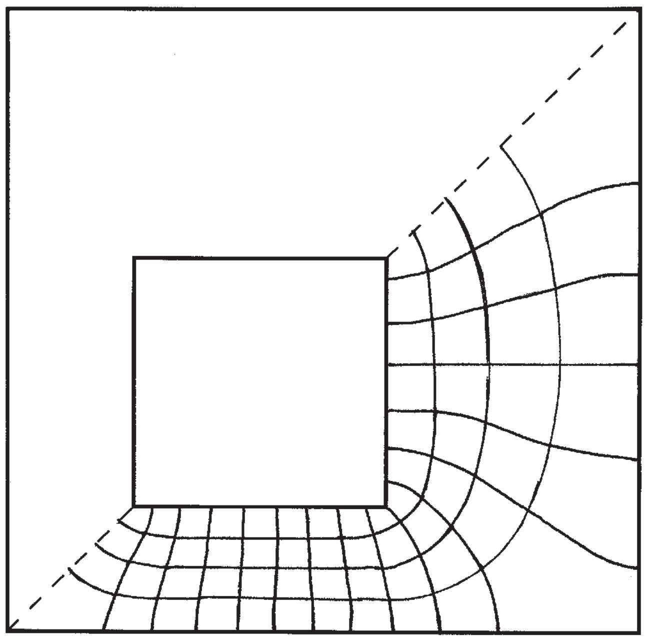


b) Assume $\epsilon = \epsilon_0$ and estimate C per meter length: In this case N_Q is the number of squares around the full perimeter of the circular conductor, or four times the number of squares shown in the drawing. N_V is the number of squares between the circle and the rectangle, or 5. The capacitance is estimated to be

$$C = \frac{N_Q}{N_V}\epsilon_0 = \frac{4 \times 13}{5}\epsilon_0 = 10.4\epsilon_0 \doteq \underline{90 \text{ pF/m}}$$

6.29. The inner conductor of the transmission line shown in Fig. 6.14 has a square cross-section $2a \times 2a$, while the outer square is $5a \times 5a$. The axes are displaced as shown. (a) Construct a good-sized drawing of the transmission line, say with $a = 2.5$ cm, and then prepare a curvilinear-square plot of the electrostatic field between the conductors. (b) Use the map to calculate the capacitance per meter length if $\epsilon = 1.6\epsilon_0$. (c) How would your result to part b change if $a = 0.6$ cm?

a) The plot is shown below. Some improvement is possible, depending on how much time one wishes to spend.



b) From the plot, the capacitance is found to be

$$C \doteq \frac{16 \times 2}{4}(1.6)\epsilon_0 = 12.8\epsilon_0 \doteq \underline{110 \text{ pF/m}}$$

c) If a is changed, the result of part b <u>would not change</u>, since all dimensions retain the same relative scale.

6.30. For the coaxial capacitor of Problem 6.18, suppose that the dielectric is leaky, allowing current to flow between the inner and outer conductors, while the electric field is still uniform with radius.

a) What functional form must the dielectric conductivity assume? We must have constant current through any cross-section, which means that $\mathbf{J} = I/(2\pi\rho)\,\mathbf{a}_\rho$ A/m^2, where I is the radial current per unit length. Then, from $\mathbf{J} = \sigma\mathbf{E}$, where $\mathbf{E}$ is constant, we require a $1/\rho$ dependence on σ, or let $\underline{\sigma = \sigma_0/\rho}$, where σ_0 is a constant.

b) What is the basic functional form of the resistance per unit distance, R? From Problem 6.18, we had $\mathbf{E} = a\rho_s/g\,\mathbf{a}_\rho$ V/m, where ρ_s is the surface charge density on the inner conductor, and g is the constant parameter in the permittivity, $\epsilon = g/\rho$. Now, $I = 2\pi\rho\sigma E = 2\pi a\rho_s\sigma_0/g$, and $V_0 = a\rho_s(b - a)/g$ (from 6.18). Then $R = V_0/I = \underline{(b - a)/(2\pi\sigma_0)}$.

6.30c) What parameters remain in the product, RC, where the form of C, the capacitance per unit distance, has been determined in Problem 6.18? With $C = 2\pi g/(b-a)$ (from 6.18), we have $\underline{RC = g/\sigma_0}$.

6.31. A two-wire transmission line consists of two parallel perfectly-conducting cylinders, each having a radius of 0.2 mm, separated by center-to-center distance of 2 mm. The medium surrounding the wires has $\epsilon_r = 3$ and $\sigma = 1.5$ mS/m. A 100-V battery is connected between the wires. Calculate:

a) the magnitude of the charge per meter length on each wire: Use

$$C = \frac{\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{\pi \times 3 \times 8.85 \times 10^{-12}}{\cosh^{-1}(1/0.2)} = 3.64 \times 10^{-9} \text{ C/m}$$

Then the charge per unit length will be

$$Q = CV_0 = (3.64 \times 10^{-11})(100) = 3.64 \times 10^{-9} \text{ C/m} = \underline{3.64\,\text{nC/m}}$$

b) the battery current: Use

$$RC = \frac{\epsilon}{\sigma} \quad \Rightarrow \quad R = \frac{3 \times 8.85 \times 10^{-12}}{(1.5 \times 10^{-3})(3.64 \times 10^{-11})} = 486\,\Omega$$

Then

$$I = \frac{V_0}{R} = \frac{100}{486} = 0.206\,\text{A} = \underline{206\,\text{mA}}$$