CHAPTER 12

12.1. Show that $E_{xs} = Ae^{jk_0 z+\phi}$ is a solution to the vector Helmholtz equation, Sec. 12.1, Eq. (30), for $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ and any ϕ and A: We take

$$\frac{d^2}{dz^2} Ae^{jk_0 z+\phi} = (jk_0)^2 Ae^{jk_0 z+\phi} = -k_0^2 E_{xs}$$

12.2. A 100-MHz uniform plane wave propagates in a lossless medium for which $\epsilon_r = 5$ and $\mu_r = 1$. Find:

a) v_p: $v_p = c/\sqrt{\epsilon_r} = 3\times 10^8/\sqrt{5} = \underline{1.34\times 10^8 \text{ m/s}}$.

b) β: $\beta = \omega/v_p = (2\pi\times 10^8)/(1.34\times 10^8) = \underline{4.69 \text{ m}^{-1}}$.

c) λ: $\lambda = 2\pi/\beta = \underline{1.34 \text{ m}}$.

d) $\mathbf{E}_s$: Assume real amplitude E_0, forward z travel, and x polarization, and write $\mathbf{E}_s = E_0 \exp(-j\beta z)\mathbf{a}_x = \underline{E_0 \exp(-j4.69z)\,\mathbf{a}_x \text{ V/m}}$.

e) $\mathbf{H}_s$: First, the intrinsic impedance of the medium is $\eta = \eta_0/\sqrt{\epsilon_r} = 377/\sqrt{5} = 169\ \Omega$. Then $\mathbf{H}_s = (E_0/\eta)\exp(-j\beta z)\,\mathbf{a}_y = \underline{(E_0/169)\exp(-j4.69z)\,\mathbf{a}_y \text{ A/m}}$.

f) $<\mathbf{S}> = (1/2)\mathcal{R}e\,\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \underline{(E_0^2/337)\,\mathbf{a}_z \text{ W/m}^2}$

12.3. An $\mathbf{H}$ field in free space is given as $\mathcal{H}(x,t) = 10\cos(10^8 t - \beta x)\mathbf{a}_y$ A/m. Find

a) β: Since we have a uniform plane wave, $\beta = \omega/c$, where we identify $\omega = 10^8$ sec^{-1}. Thus $\beta = 10^8/(3\times 10^8) = \underline{0.33 \text{ rad/m}}$.

b) λ: We know $\lambda = 2\pi/\beta = \underline{18.9 \text{ m}}$.

c) $\mathcal{E}(x,t)$ at $P(0.1, 0.2, 0.3)$ at $t = 1$ ns: Use $E(x,t) = -\eta_0 H(x,t) = -(377)(10)\cos(10^8 t - \beta x) = -3.77\times 10^3\cos(10^8 t - \beta x)$. The vector direction of $\mathbf{E}$ will be $-\mathbf{a}_z$, since we require that $\mathbf{S} = \mathbf{E}\times\mathbf{H}$, where $\mathbf{S}$ is x-directed. At the given point, the relevant coordinate is $x = 0.1$. Using this, along with $t = 10^{-9}$ sec, we finally obtain

$$\mathbf{E}(x,t) = -3.77\times 10^3\cos[(10^8)(10^{-9}) - (0.33)(0.1)]\mathbf{a}_z = -3.77\times 10^3\cos(6.7\times 10^{-2})\mathbf{a}_z$$
$$= \underline{-3.76\times 10^3\mathbf{a}_z \text{ V/m}}$$

12.4. Given $\mathcal{E}(z,t) = E_0 e^{-\alpha z}\sin(\omega t - \beta z)\mathbf{a}_x$, and $\eta = |\eta|e^{j\phi}$, find:

a) $\mathbf{E}_s$: Using the Euler identity for the sine, we can write the given field in the form:

$$\mathcal{E}(z,t) = E_0\,e^{-\alpha z}\left[\frac{e^{j(\omega t-\beta z)} - e^{-j(\omega t-\beta z)}}{2j}\right]\mathbf{a}_x = -\frac{jE_0}{2}e^{-\alpha z}e^{j(\omega t-\beta z)}\mathbf{a}_x + c.c.$$

We therefore identify the phasor form as $\mathbf{E}_s(z) = \underline{-jE_0 e^{-\alpha z}e^{-j\beta z}\,\mathbf{a}_x \text{ V/m}}$.

b) $\mathbf{H}_s$: With positive z travel, and with $\mathbf{E}_s$ along positive x, $\mathbf{H}_s$ will lie along positive y. Therefore $\mathbf{H}_s = \underline{-jE_0/|\eta|\,e^{-\alpha z}e^{-j\beta z}e^{-j\phi}\,\mathbf{a}_y \text{ A/m}}$.

c) $<\mathbf{S}>$:

$$<\mathbf{S}> = (1/2)\mathcal{R}e\{\mathbf{E}_s\times\mathbf{H}_s^*\} = \underline{\frac{E_0^2}{2|\eta|}\,e^{-2\alpha z}\cos\phi\,\mathbf{a}_z \text{ W/m}^2}$$

12.5. A 150-MHz uniform plane wave in free space is described by $\mathbf{H}_s = (4 + j10)(2\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z}$ A/m.

a) Find numerical values for ω, λ, and β: First, $\omega = 2\pi \times 150 \times 10^6 = \underline{3\pi \times 10^8 \text{ sec}^{-1}}$. Second, for a uniform plane wave in free space, $\lambda = 2\pi c/\omega = c/f = (3 \times 10^8)/(1.5 \times 10^8) = \underline{2\,\text{m}}$. Third, $\beta = 2\pi/\lambda = \underline{\pi\,\text{rad/m}}$.

b) Find $\mathcal{H}(z,t)$ at $t = 1.5$ ns, $z = 20$ cm: Use

$$\begin{aligned}\mathbf{H}(z,t) &= \text{Re}\{\mathbf{H}_s e^{j\omega t}\} = \text{Re}\{(4 + j10)(2\mathbf{a}_x + j\mathbf{a}_y)(\cos(\omega t - \beta z) + j\sin(\omega t - \beta z)\} \\ &= [8\cos(\omega t - \beta z) - 20\sin(\omega t - \beta z)]\,\mathbf{a}_x - [10\cos(\omega t - \beta z) + 4\sin(\omega t - \beta z)]\,\mathbf{a}_y\end{aligned}$$

. Now at the given position and time, $\omega t - \beta z = (3\pi \times 10^8)(1.5 \times 10^{-9}) - \pi(0.20) = \pi/4$. And $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$. So finally,

$$\mathbf{H}(z = 20\text{cm}, t = 1.5\text{ns}) = -\frac{1}{\sqrt{2}}(12\mathbf{a}_x + 14\mathbf{a}_y) = \underline{-8.5\mathbf{a}_x - 9.9\mathbf{a}_y \text{ A/m}}$$

c) What is $|E|_{max}$? Have $|E|_{max} = \eta_0 |H|_{max}$, where

$$|H|_{max} = \sqrt{\mathbf{H}_s \cdot \mathbf{H}_s^*} = [4(4 + j10)(4 - j10) + (j)(-j)(4 + j10)(4 - j10)]^{1/2} = 24.1\,\text{A/m}$$

Then $|E|_{max} = 377(24.1) = \underline{9.08 \text{ kV/m}}$.

12.6. A linearly-polarized plane wave in free space has electric field given by $\mathcal{E}(z,t) = (25\mathbf{a}_x - 30\mathbf{a}_z)\cos(\omega t - 50y)$ V/m. Find:

a) ω: In free space, $\beta = k_0 = \omega/c \;\Rightarrow\; \omega = 50c = 50 \times 3 \times 10^8 = \underline{1.5 \times 10^{10} \text{ rad/s}}$.

b) $\mathbf{E}_s = \underline{(25\mathbf{a}_x - 30\mathbf{a}_z)\exp(-j50y) \text{ V/m}}$.

c) $\mathbf{H}_s$: We use the fact that each to component of $\mathbf{E}_s$, there will be an orthogonal $\mathbf{H}_s$ component, oriented such that the cross product of $\mathbf{E}_s$ with $\mathbf{H}_s$ gives the propagation direction. We obtain

$$\mathbf{H}_s = -\frac{1}{\eta_0}(25\mathbf{a}_z + 30\mathbf{a}_x)\,e^{-j50y}$$

d)

$$\begin{aligned}<\mathbf{S}> &= \frac{1}{2}\mathcal{R}e\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{1}{2\eta_0}\mathcal{R}e\left\{(25\mathbf{a}_x - 30\mathbf{a}_z) \times (-25\mathbf{a}_z - 30\mathbf{a}_x)\right\} \\ &= \frac{1}{2(377)}\left[(25)^2 + (30)^2\right]\mathbf{a}_y = \underline{2.0\,\mathbf{a}_y \text{ W/m}^2}\end{aligned}$$

12.7. The phasor magnetic field intensity for a 400-MHz uniform plane wave propagating in a certain lossless material is $(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x}$ A/m. Knowing that the maximum amplitude of $\mathbf{E}$ is 1500 V/m, find β, η, λ, v_p, ϵ_r, μ_r, and $\mathcal{H}(x,y,z,t)$: First, from the phasor expression, we identify $\beta = \underline{25 \text{ m}^{-1}}$ from the argument of the exponential function. Next, we evaluate $H_0 = |\mathbf{H}| = \sqrt{\mathbf{H} \cdot \mathbf{H}^*} = \sqrt{2^2 + 5^2} = \sqrt{29}$. Then $\eta = E_0/H_0 = 1500/\sqrt{29} = \underline{278.5\ \Omega}$. Then $\lambda = 2\pi/\beta = 2\pi/25 = .25 \text{ m} = \underline{25 \text{ cm}}$. Next,

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 400 \times 10^6}{25} = \underline{1.01 \times 10^8 \text{ m/s}}$$

12.7. (continued) Now we note that

$$\eta = 278.5 = 377\sqrt{\frac{\mu_r}{\epsilon_r}} \quad \Rightarrow \quad \frac{\mu_r}{\epsilon_r} = 0.546$$

And

$$v_p = 1.01 \times 10^8 = \frac{c}{\sqrt{\mu_r \epsilon_r}} \quad \Rightarrow \quad \mu_r \epsilon_r = 8.79$$

We solve the above two equations simultaneously to find $\epsilon_r = \underline{4.01}$ and $\mu_r = \underline{2.19}$. Finally,

$$\begin{aligned}\mathbf{H}(x,y,z,t) &= Re\left\{(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x}e^{j\omega t}\right\} \\ &= 2\cos(2\pi \times 400 \times 10^6 t - 25x)\mathbf{a}_y + 5\sin(2\pi \times 400 \times 10^6 t - 25x)\mathbf{a}_z \\ &= \underline{2\cos(8\pi \times 10^8 t - 25x)\mathbf{a}_y + 5\sin(8\pi \times 10^8 t - 25x)\mathbf{a}_z \text{ A/m}}\end{aligned}$$

12.8. Let the fields, $\mathcal{E}(z,t) = 1800\cos(10^7\pi t - \beta z)\mathbf{a}_x$ V/m and $\mathcal{H}(z,t) = 3.8\cos(10^7\pi t - \beta z)\mathbf{a}_y$ A/m, represent a uniform plane wave propagating at a velocity of 1.4×10^8 m/s in a perfect dielectric. Find:

a) $\beta = \omega/v = (10^7\pi)/(1.4 \times 10^8) = \underline{0.224\,\text{m}^{-1}}$.

b) $\lambda = 2\pi/\beta = 2\pi/.224 = \underline{28.0\,\text{m}}$.

c) $\eta = |\mathbf{E}|/|\mathbf{H}| = 1800/3.8 = \underline{474\,\Omega}$.

d) μ_r: Have two equations in the two unknowns, μ_r and ϵ_r: $\eta = \eta_0\sqrt{\mu_r/\epsilon_r}$ and $\beta = \omega\sqrt{\mu_r\epsilon_r}/c$. Eliminate ϵ_r to find

$$\mu_r = \left[\frac{\beta c \eta}{\omega \eta_0}\right]^2 = \left[\frac{(.224)(3 \times 10^8)(474)}{(10^7\pi)(377)}\right]^2 = \underline{2.69}$$

e) $\epsilon_r = \mu_r(\eta_0/\eta)^2 = (2.69)(377/474)^2 = \underline{1.70}$.

12.9. A certain lossless material has $\mu_r = 4$ and $\epsilon_r = 9$. A 10-MHz uniform plane wave is propagating in the $\mathbf{a}_y$ direction with $E_{x0} = 400$ V/m and $E_{y0} = E_{z0} = 0$ at $P(0.6, 0.6, 0.6)$ at $t = 60$ ns.

a) Find β, λ, v_p, and η: For a uniform plane wave,

$$\beta = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}\sqrt{\mu_r\epsilon_r} = \frac{2\pi \times 10^7}{3 \times 10^8}\sqrt{(4)(9)} = \underline{0.4\pi \text{ rad/m}}$$

Then $\lambda = (2\pi)/\beta = (2\pi)/(0.4\pi) = \underline{5\,\text{m}}$. Next,

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 10^7}{4\pi \times 10^{-1}} = \underline{5 \times 10^7 \text{ m/s}}$$

Finally,

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \eta_0\sqrt{\frac{\mu_r}{\epsilon_r}} = 377\sqrt{\frac{4}{9}} = \underline{251\ \Omega}$$

b) Find $E(t)$ (at P): We are given the amplitude at $t = 60$ ns and at $y = 0.6$ m. Let the maximum amplitude be E_{max}, so that in general, $E_x = E_{max}\cos(\omega t - \beta y)$. At the given position and time,

$$E_x = 400 = E_{max}\cos[(2\pi \times 10^7)(60 \times 10^{-9}) - (4\pi \times 10^{-1})(0.6)] = E_{max}\cos(0.96\pi)$$
$$= -0.99E_{max}$$

So $E_{max} = (400)/(-0.99) = -403\,\text{V/m}$. Thus at P, $E(t) = \underline{-403\cos(2\pi \times 10^7 t)\ \text{V/m}}$.

c) Find $H(t)$: First, we note that if E at a given instant points in the negative x direction, while the wave propagates in the forward y direction, then H at that same position and time must point in the positive z direction. Since we have a lossless homogeneous medium, η is real, and we are allowed to write $H(t) = E(t)/\eta$, where η is treated as negative and real. Thus

$$H(t) = H_z(t) = \frac{E_x(t)}{\eta} = \frac{-403}{-251}\cos(2\pi \times 10^{-7}t) = \underline{1.61\cos(2\pi \times 10^{-7}t)\ \text{A/m}}$$

12.10. In a medium characterized by intrinsic impedance $\eta = |\eta|e^{j\phi}$, a linearly-polarized plane wave propagates, with magnetic field given as $\mathbf{H}_s = (H_{0y}\mathbf{a}_y + H_{0z}\mathbf{a}_z)\,e^{-\alpha x}e^{-j\beta x}$. Find:

a) $\mathbf{E}_s$: Requiring orthogonal components of $\mathbf{E}_s$ for each component of $\mathbf{H}_s$, we find

$$\mathbf{E}_s = |\eta|\,[H_{0z}\,\mathbf{a}_y - H_{0y}\,\mathbf{a}_z]\ e^{-\alpha x}\ e^{-j\beta x}\ e^{j\phi}$$

b) $\mathcal{E}(x,t) = \mathcal{R}e\,\{\mathbf{E}_s e^{j\omega t}\} = |\eta|\,[H_{0z}\,\mathbf{a}_y - H_{0y}\,\mathbf{a}_z]\,e^{-\alpha x}\ \cos(\omega t - \beta x + \phi)$.

c) $\mathcal{H}(x,t) = \mathcal{R}e\,\{\mathbf{H}_s e^{j\omega t}\} = [H_{0y}\,\mathbf{a}_y + H_{0z}\,\mathbf{a}_z]\,e^{-\alpha x}\ \cos(\omega t - \beta x)$.

d) $$< \mathbf{S} >= \frac{1}{2}\,\mathcal{R}e\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{1}{2}|\eta|\left[H_{0y}^2 + H_{0z}^2\right]e^{-2\alpha x}\ \cos\phi\,\mathbf{a}_x\ \text{W/m}^2$$

12.11. A 2-GHz uniform plane wave has an amplitude of $E_{y0} = 1.4$ kV/m at $(0,0,0,t=0)$ and is propagating in the $\mathbf{a}_z$ direction in a medium where $\epsilon'' = 1.6 \times 10^{-11}$ F/m, $\epsilon' = 3.0 \times 10^{-11}$ F/m, and $\mu = 2.5\,\mu$H/m. Find:

a) E_y at $P(0,0,1.8\text{cm})$ at 0.2 ns: To begin, we have the ratio, $\epsilon''/\epsilon' = 1.6/3.0 = 0.533$. So

$$\alpha = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]^{1/2}$$
$$= (2\pi \times 2 \times 10^9)\sqrt{\frac{(2.5 \times 10^{-6})(3.0 \times 10^{-11})}{2}}\left[\sqrt{1+(.533)^2} - 1\right]^{1/2} = 28.1\,\text{Np/m}$$

Then

$$\beta = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]^{1/2} = 112\,\text{rad/m}$$

Thus in general,

$$E_y(z,t) = 1.4e^{-28.1z}\cos(4\pi \times 10^9 t - 112z)\ \text{kV/m}$$

12.11a. (continued) Evaluating this at $t = 0.2$ ns and $z = 1.8$ cm, find

$$E_y(1.8\,\text{cm}, 0.2\,\text{ns}) = \underline{0.74 \text{ kV/m}}$$

b) H_x at P at 0.2 ns: We use the phasor relation, $H_{xs} = -E_{ys}/\eta$ where

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}} = \sqrt{\frac{2.5 \times 10^{-6}}{3.0 \times 10^{-11}}}\frac{1}{\sqrt{1 - j(.533)}} = 263 + j65.7 = 271\angle\, 14^\circ\ \Omega$$

So now

$$H_{xs} = -\frac{E_{ys}}{\eta} = -\frac{(1.4 \times 10^3)e^{-28.1z}e^{-j112z}}{271e^{j14^\circ}} = -5.16e^{-28.1z}e^{-j112z}e^{-j14^\circ}\ \text{A/m}$$

Then

$$H_x(z,t) = -5.16e^{-28.1z}\cos(4\pi \times 10^{-9}t - 112z - 14^\circ)$$

This, when evaluated at $t = 0.2$ ns and $z = 1.8$ cm, yields

$$H_x(1.8\,\text{cm}, 0.2\,\text{ns}) = \underline{-3.0\,\text{A/m}}$$

12.12. The plane wave $\mathbf{E}_s = 300e^{-jkx}\mathbf{a}_y$ V/m is propagating in a material for which $\mu = 2.25\ \mu$H/m, $\epsilon' = 9$ pF/m, and $\epsilon'' = 7.8$ pF/m. If $\omega = 64$ Mrad/s, find:

a) α: We use the general formula, Eq. (35):

$$\alpha = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]^{1/2}$$

$$= (64 \times 10^6)\sqrt{\frac{(2.25 \times 10^{-6})(9 \times 10^{-12})}{2}}\left[\sqrt{1 + (.867)^2} - 1\right]^{1/2} = \underline{0.116 \text{ Np/m}}$$

b) β: Using (36), we write

$$\beta = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]^{1/2} = \underline{.311 \text{ rad/m}}$$

c) $v_p = \omega/\beta = (64 \times 10^6)/(.311) = \underline{2.06 \times 10^8 \text{ m/s}}$.

d) $\lambda = 2\pi/\beta = 2\pi/(.311) = \underline{20.2 \text{ m}}$.

e) η: Using (39):

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}} = \sqrt{\frac{2.25 \times 10^{-6}}{9 \times 10^{-12}}}\frac{1}{\sqrt{1 - j(.867)}} = 407 + j152 = \underline{434.5e^{j.36}\ \Omega}$$

f) $\mathbf{H}_s$: With $\mathbf{E}_s$ in the positive y direction (at a given time) and propagating in the positive x direction, we would have a positive z component of $\mathbf{H}_s$, at the same time. We write (with $jk = \alpha + j\beta$):

$$\mathbf{H}_s = \frac{E_s}{\eta}\mathbf{a}_z = \frac{300}{434.5e^{j.36}}e^{-jkx}\mathbf{a}_z = 0.69e^{-\alpha x}e^{-j\beta x}e^{-j.36}\mathbf{a}_z$$

$$= \underline{0.69e^{-.116x}e^{-j.311x}e^{-j.36}\mathbf{a}_z \text{ A/m}}$$

2.12g) $\mathcal{E}(3,2,4,10\text{ns})$: The real instantaneous form of $\mathbf{E}$ will be

$$\mathbf{E}(x,y,z,t) = \text{Re}\left\{\mathbf{E}_s e^{j\omega t}\right\} = 300e^{-\alpha x}\cos(\omega t - \beta x)\mathbf{a}_y$$

Therefore

$$\mathbf{E}(3,2,4,10\text{ns}) = 300e^{-.116(3)}\cos[(64\times 10^6)(10^{-8}) - .311(3)]\mathbf{a}_y = \underline{203 \text{ V/m}}$$

12.13. Let $jk = 0.2 + j1.5\,\text{m}^{-1}$ and $\eta = 450 + j60\ \Omega$ for a uniform plane wave propagating in the $\mathbf{a}_z$ direction. If $\omega = 300$ Mrad/s, find μ, ϵ', and ϵ'': We begin with

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}} = 450 + j60$$

and

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j(\epsilon''/\epsilon')} = 0.2 + j1.5$$

Then

$$\eta\eta^* = \frac{\mu}{\epsilon'}\frac{1}{\sqrt{1 + (\epsilon''/\epsilon')^2}} = (450 + j60)(450 - j60) = 2.06\times 10^5 \tag{1}$$

and

$$(jk)(jk)^* = \omega^2\mu\epsilon'\sqrt{1 + (\epsilon''/\epsilon')^2} = (0.2 + j1.5)(0.2 - j1.5) = 2.29 \tag{2}$$

Taking the ratio of (2) to (1),

$$\frac{(jk)(jk)^*}{\eta\eta^*} = \omega^2(\epsilon')^2\left(1 + (\epsilon''/\epsilon')^2\right) = \frac{2.29}{2.06\times 10^5} = 1.11\times 10^{-5}$$

Then with $\omega = 3\times 10^8$,

$$(\epsilon')^2 = \frac{1.11\times 10^{-5}}{(3\times 10^8)^2\,(1 + (\epsilon''/\epsilon')^2)} = \frac{1.23\times 10^{-22}}{(1 + (\epsilon''/\epsilon')^2)} \tag{3}$$

Now, we use Eqs. (35) and (36). Squaring these and taking their ratio gives

$$\frac{\alpha^2}{\beta^2} = \frac{\sqrt{1 + (\epsilon''/\epsilon')^2}}{\sqrt{1 + (\epsilon''/\epsilon')^2}} = \frac{(0.2)^2}{(1.5)^2}$$

We solve this to find $\epsilon''/\epsilon' = 0.271$. Substituting this result into (3) gives $\epsilon' = 1.07\times 10^{-11}$ F/m. Since $\epsilon''/\epsilon' = 0.271$, we then find $\epsilon'' = 2.90\times 10^{-12}$ F/m. Finally, using these results in either (1) or (2) we find $\mu = 2.28\times 10^{-6}$ H/m. Summary: $\mu = \underline{2.28\times 10^{-6}\text{ H/m}}$, $\epsilon' = \underline{1.07\times 10^{-11}\text{ F/m}}$, and $\epsilon'' = \underline{2.90\times 10^{-12}\text{ F/m}}$.

12.14. A certain nonmagnetic material has the material constants $\epsilon_r' = 2$ and $\epsilon''/\epsilon' = 4 \times 10^{-4}$ at $\omega = 1.5$ Grad/s. Find the distance a uniform plane wave can propagate through the material before:

a) it is attenuated by 1 Np: First, $\epsilon'' = (4 \times 10^4)(2)(8.854 \times 10^{-12}) = 7.1 \times 10^{-15}$ F/m. Then, since $\epsilon''/\epsilon' << 1$, we use the approximate form for α, given by Eq. (51) (written in terms of ϵ''):

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{(1.5 \times 10^9)(7.1 \times 10^{-15})}{2}\frac{377}{\sqrt{2}} = 1.42 \times 10^{-3} \text{ Np/m}$$

The required distance is now $z_1 = (1.42 \times 10^{-3})^{-1} = \underline{706 \text{ m}}$

b) the power level is reduced by one-half: The governing relation is $e^{-2\alpha z_{1/2}} = 1/2$, or $z_{1/2} = \ln 2/2\alpha = \ln 2/2(1.42 \times 10^-3) = \underline{244 \text{ m}}$.

c) the phase shifts 360°: This distance is defined as one wavelength, where $\lambda = 2\pi/\beta$ $= (2\pi c)/(\omega\sqrt{\epsilon_r'}) = [2\pi(3 \times 10^8)]/[(1.5 \times 10^9)\sqrt{2}] = \underline{0.89 \text{ m}}$.

12.15. A 10 GHz radar signal may be represented as a uniform plane wave in a sufficiently small region. Calculate the wavelength in centimeters and the attenuation in nepers per meter if the wave is propagating in a non-magnetic material for which

a) $\epsilon_r' = 1$ and $\epsilon_r'' = 0$: In a non-magnetic material, we would have:

$$\alpha = \omega\sqrt{\frac{\mu_0\epsilon_0\epsilon_r'}{2}}\left[\sqrt{1+\left(\frac{\epsilon_r''}{\epsilon_r'}\right)^2} - 1\right]^{1/2}$$

and

$$\beta = \omega\sqrt{\frac{\mu_0\epsilon_0\epsilon_r'}{2}}\left[\sqrt{1+\left(\frac{\epsilon_r''}{\epsilon_r'}\right)^2} + 1\right]^{1/2}$$

With the given values of ϵ_r' and ϵ_r'', it is clear that $\beta = \omega\sqrt{\mu_0\epsilon_0} = \omega/c$, and so

$\lambda = 2\pi/\beta = 2\pi c/\omega = 3 \times 10^{10}/10^{10} = \underline{3 \text{ cm}}$. It is also clear that $\alpha = \underline{0}$.

b) $\epsilon_r' = 1.04$ and $\epsilon_r'' = 9.00 \times 10^{-4}$: In this case $\epsilon_r''/\epsilon_r' << 1$, and so $\beta \doteq \omega\sqrt{\epsilon_r'}/c = 2.13 \text{ cm}^{-1}$. Thus $\lambda = 2\pi/\beta = \underline{2.95 \text{ cm}}$. Then

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\omega\epsilon_r''}{2}\frac{\sqrt{\mu_0\epsilon_0}}{\sqrt{\epsilon_r'}} = \frac{\omega}{2c}\frac{\epsilon_r''}{\sqrt{\epsilon_r'}} = \frac{2\pi \times 10^{10}}{2 \times 3 \times 10^8}\frac{(9.00 \times 10^{-4})}{\sqrt{1.04}}$$
$$= \underline{9.24 \times 10^{-2} \text{ Np/m}}$$

2.15c) $\epsilon'_r = 2.5$ and $\epsilon''_r = 7.2$: Using the above formulas, we obtain

$$\beta = \frac{2\pi \times 10^{10}\sqrt{2.5}}{(3\times 10^{10})\sqrt{2}}\left[\sqrt{1+\left(\frac{7.2}{2.5}\right)^2}+1\right]^{1/2} = 4.71\ \text{cm}^{-1}$$

and so $\lambda = 2\pi/\beta = \underline{1.33\ \text{cm}}$. Then

$$\alpha = \frac{2\pi \times 10^{10}\sqrt{2.5}}{(3\times 10^{8})\sqrt{2}}\left[\sqrt{1+\left(\frac{7.2}{2.5}\right)^2}-1\right]^{1/2} = \underline{335\ \text{Np/m}}$$

12.16. The power factor of a capacitor is defined as the cosine of the impedance phase angle, and its Q is ωCR, where R is the parallel resistance. Assume an idealized parallel plate capacitor having a dielecric characterized by σ, ϵ', and μ_r. Find both the power factor and Q in terms of the loss tangent: First, the impedance will be:

$$Z = \frac{R\left(\frac{1}{j\omega C}\right)}{R+\left(\frac{1}{j\omega C}\right)} = R\frac{1-jR\omega C}{1+(R\omega C)^2} = R\frac{1-jQ}{1+Q^2}$$

Now $R = d/(\sigma A)$ and $C = \epsilon' A/d$, and so $Q = \omega\epsilon'/\sigma = \underline{1/l.t.}$ Then the power factor is P.F $= \cos[\tan^{-1}(-Q)] = \underline{1/\sqrt{1+Q^2}}$.

12.17. Let $\eta = 250 + j30\,\Omega$ and $jk = 0.2 + j2\,\text{m}^{-1}$ for a uniform plane wave propagating in the $\mathbf{a}_z$ direction in a dielectric having some finite conductivity. If $|E_s| = 400$ V/m at $z = 0$, find:

a) $<\mathbf{S}>$ at $z = 0$ and $z = 60$ cm: Assume x-polarization for the electric field. Then

$$\begin{aligned} <\mathbf{S}> &= \frac{1}{2}\text{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\text{Re}\left\{400e^{-\alpha z}e^{-j\beta z}\mathbf{a}_x \times \frac{400}{\eta^*}e^{-\alpha z}e^{j\beta z}\mathbf{a}_y\right\} \\ &= \frac{1}{2}(400)^2 e^{-2\alpha z}\text{Re}\left\{\frac{1}{\eta^*}\right\}\mathbf{a}_z = 8.0\times 10^4 e^{-2(0.2)z}\text{Re}\left\{\frac{1}{250-j30}\right\}\mathbf{a}_z \\ &= 315\, e^{-2(0.2)z}\,\mathbf{a}_z\ \text{W/m}^2 \end{aligned}$$

Evaluating at $z = 0$, obtain $<\mathbf{S}>(z=0) = \underline{315\,\mathbf{a}_z\ \text{W/m}^2}$,
and at $z = 60$ cm, $\mathbf{P}_{z,av}(z = 0.6) = 315e^{-2(0.2)(0.6)}\mathbf{a}_z = \underline{248\,\mathbf{a}_z\ \text{W/m}^2}$.

b) the average ohmic power dissipation in watts per cubic meter at $z = 60$ cm: At this point a flaw becomes evident in the problem statement, since solving this part in two different ways gives results that are not the same. I will demonstrate: In the first method, we use Poynting's theorem in point form (first equation at the top of p. 366), which we modify for the case of time-average fields to read:

$$-\nabla\cdot <\mathbf{S}> = <\mathbf{J}\cdot\mathbf{E}>$$

where the right hand side is the average power dissipation per volume. Note that the additional right-hand-side terms in Poynting's theorem that describe changes in energy stored in the fields will both be zero in steady state. We apply our equation to the result of part a:

$$< \mathbf{J} \cdot \mathbf{E} >= -\nabla \cdot < \mathbf{S} >= -\frac{d}{dz} 315\, e^{-2(0.2)z} = (0.4)(315)e^{-2(0.2)z} = 126e^{-0.4z} \text{ W/m}^3$$

At $z = 60$ cm, this becomes $< \mathbf{J} \cdot \mathbf{E} >= 99.1$ W/m^3. In the second method, we solve for the conductivity and evaluate $< \mathbf{J} \cdot \mathbf{E} >= \sigma < E^2 >$. We use

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j(\epsilon''/\epsilon')}$$

and

$$\eta = \sqrt{\frac{\mu}{\epsilon'}} \frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}}$$

We take the ratio,

$$\frac{jk}{\eta} = j\omega\epsilon' \left[1 - j\left(\frac{\epsilon''}{\epsilon'}\right)\right] = j\omega\epsilon' + \omega\epsilon''$$

Identifying $\sigma = \omega\epsilon''$, we find

$$\sigma = \text{Re}\left\{\frac{jk}{\eta}\right\} = \text{Re}\left\{\frac{0.2 + j2}{250 + j30}\right\} = 1.74 \times 10^{-3} \text{ S/m}$$

Now we find the dissipated power per volume:

$$\sigma < E^2 >= 1.74 \times 10^{-3} \left(\frac{1}{2}\right) \left(400e^{-0.2z}\right)^2$$

At $z = 60$ cm, this evaluates as 109 W/m^3. One can show that consistency between the two methods requires that

$$\text{Re}\left\{\frac{1}{\eta^*}\right\} = \frac{\sigma}{2\alpha}$$

This relation does not hold using the numbers as given in the problem statement and the value of σ found above. Note that in Problem 12.13, where all values are worked out, the relation does hold and consistent results are obtained using both methods.

12.18. Given, a 100MHz uniform plane wave in a medium known to be a good dielectric. The phasor electric field is $\mathbf{E}_s = 4e^{-0.5z}e^{-j20z}\mathbf{a}_x$ V/m. Not stated in the problem is the permeability, which we take to be μ_0. *Also, the specified distance in part f should be 10m, not 1km.* Determine:

a) ϵ': As a first step, it is useful to see just how much of a good dielectric we have. We use the good dielectric approximations, Eqs. (60a) and (60b), with $\sigma = \omega\epsilon''$. Using these, we take the ratio, β/α, to find

$$\frac{\beta}{\alpha} = \frac{20}{0.5} = \frac{\omega\sqrt{\mu\epsilon'}\left[1 + (1/8)(\epsilon''/\epsilon')^2\right]}{(\omega\epsilon''/2)\sqrt{\mu/\epsilon'}} = 2\left(\frac{\epsilon'}{\epsilon''}\right) + \frac{1}{4}\left(\frac{\epsilon''}{\epsilon'}\right)$$

This becomes the quadratic equation:

$$\left(\frac{\epsilon''}{\epsilon'}\right)^2 - 160\left(\frac{\epsilon''}{\epsilon'}\right) + 8 = 0$$

12.18a (continued) The solution to the quadratic is $(\epsilon''/\epsilon') = 0.05$, which means that we can neglect the second term in Eq. (60b), so that $\beta \doteq \omega\sqrt{\mu\epsilon'} = (\omega/c)\sqrt{\epsilon_r'}$. With the given frequency of 100 MHz, and with $\mu = \mu_0$, we find $\sqrt{\epsilon_r'} = 20(3/2\pi) = 9.55$, so that $\epsilon_r' = 91.3$, and finally $\epsilon' = \epsilon_r'\epsilon_0 = \underline{8.1 \times 10^{-10}\ \text{F/m}}$.

b) ϵ'': Using Eq. (60a), the set up is

$$\alpha = 0.5 = \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} \quad \Rightarrow \quad \epsilon'' = \frac{2(0.5)}{2\pi \times 10^8}\sqrt{\frac{\epsilon'}{\mu}} = \frac{10^{-8}}{2\pi(377)}\sqrt{91.3} = \underline{4.0 \times 10^{-11}\ \text{F/m}}$$

c) η: Using Eq. (62b), we find

$$\eta \doteq \sqrt{\frac{\mu}{\epsilon'}}\left[1 + j\frac{1}{2}\left(\frac{\epsilon''}{\epsilon'}\right)\right] = \frac{377}{\sqrt{91.3}}(1 + j.025) = \underline{(39.5 + j0.99)\ \text{ohms}}$$

d) $\mathbf{H}_s$: This will be a y-directed field, and will be

$$\mathbf{H}_s = \frac{E_s}{\eta}\mathbf{a}_y = \frac{4}{(39.5 + j0.99)}e^{-0.5z}e^{-j20z}\,\mathbf{a}_y = \underline{0.101e^{-0.5z}e^{-j20z}e^{-j0.025}\,\mathbf{a}_y\ \text{A/m}}$$

e) $< \mathbf{S} >$: Using the given field and the result of part d, obtain

$$< \mathbf{S} > = \frac{1}{2}\mathcal{R}e\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{(0.101)(4)}{2}e^{-2(0.5)z}\cos(0.025)\,\mathbf{a}_z = \underline{0.202e^{-z}\,\mathbf{a}_z\ \text{W/m}^2}$$

f) the power in watts that is incident on a rectangular surface measuring 20m x 30m at $z = 10$m (not 1km): At 10m, the power density is $< \mathbf{S} > = 0.202e^{-10} = 9.2 \times 10^{-6}\ \text{W/m}^2$. The incident power on the given area is then $P = 9.2 \times 10^{-6} \times (20)(30) = \underline{5.5\ \text{mW}}$.

12.19. Perfectly-conducting cylinders with radii of 8 mm and 20 mm are coaxial. The region between the cylinders is filled with a perfect dielectric for which $\epsilon = 10^{-9}/4\pi$ F/m and $\mu_r = 1$. If $\mathbf{E}$ in this region is $(500/\rho)\cos(\omega t - 4z)\mathbf{a}_\rho$ V/m, find:

a) ω, with the help of Maxwell's equations in cylindrical coordinates: We use the two curl equations, beginning with $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$, where in this case,

$$\nabla \times \mathbf{E} = \frac{\partial E_\rho}{\partial z}\mathbf{a}_\phi = \frac{2000}{\rho}\sin(\omega t - 4z)\mathbf{a}_\phi = -\frac{\partial B_\phi}{\partial t}\mathbf{a}_\phi$$

So

$$B_\phi = \int \frac{2000}{\rho}\sin(\omega t - 4z)dt = \frac{2000}{\omega\rho}\cos(\omega t - 4z)\ \text{T}$$

Then

$$H_\phi = \frac{B_\phi}{\mu_0} = \frac{2000}{(4\pi \times 10^{-7})\omega\rho}\cos(\omega t - 4z)\ \text{A/m}$$

We next use $\nabla \times \mathbf{H} = \partial\mathbf{D}/\partial t$, where in this case

$$\nabla \times \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho}\mathbf{a}_z$$

where the second term on the right hand side becomes zero when substituting our H_ϕ. So

$$\nabla \times \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho = -\frac{8000}{(4\pi \times 10^{-7})\omega\rho}\sin(\omega t - 4z)\mathbf{a}_\rho = \frac{\partial D_\rho}{\partial t}\mathbf{a}_\rho$$

And

$$D_\rho = \int -\frac{8000}{(4\pi \times 10^{-7})\omega\rho}\sin(\omega t - 4z)dt = \frac{8000}{(4\pi \times 10^{-7})\omega^2\rho}\cos(\omega t - 4z)\ \text{C/m}^2$$

12.19a. (continued) Finally, using the given ϵ,

$$E_\rho = \frac{D_\rho}{\epsilon} = \frac{8000}{(10^{-16})\omega^2\rho}\cos(\omega t - 4z)\ \text{V/m}$$

This must be the same as the given field, so we require

$$\frac{8000}{(10^{-16})\omega^2\rho} = \frac{500}{\rho} \quad \Rightarrow \quad \omega = \underline{4 \times 10^8\ \text{rad/s}}$$

b) $\mathbf{H}(\rho, z, t)$: From part a, we have

$$\mathbf{H}(\rho, z, t) = \frac{2000}{(4\pi \times 10^{-7})\omega\rho}\cos(\omega t - 4z)\mathbf{a}_\phi = \underline{\frac{4.0}{\rho}\cos(4 \times 10^8 t - 4z)\mathbf{a}_\phi\ \text{A/m}}$$

c) $\mathbf{S}(\rho, \phi, z)$: This will be

$$\mathbf{S}(\rho, \phi, z) = \mathbf{E} \times \mathbf{H} = \frac{500}{\rho}\cos(4 \times 10^8 t - 4z)\mathbf{a}_\rho \times \frac{4.0}{\rho}\cos(4 \times 10^8 t - 4z)\mathbf{a}_\phi$$
$$= \underline{\frac{2.0 \times 10^{-3}}{\rho^2}\cos^2(4 \times 10^8 t - 4z)\mathbf{a}_z\ \text{W/m}^2}$$

d) the average power passing through every cross-section $8 < \rho < 20$ mm, $0 < \phi < 2\pi$. Using the result of part c, we find $< \mathbf{S} >= (1.0 \times 10^3)/\rho^2 \mathbf{a}_z$ W/m^2. The power through the given cross-section is now

$$\text{P} = \int_0^{2\pi}\int_{.008}^{.020} \frac{1.0 \times 10^3}{\rho^2}\rho\, d\rho\, d\phi = 2\pi \times 10^3 \ln\left(\frac{20}{8}\right) = \underline{5.7\,\text{kW}}$$

12.20. If $\mathbf{E}_s = (60/r)\sin\theta\, e^{-j2r}\,\mathbf{a}_\theta$ V/m, and $\mathbf{H}_s = (1/4\pi r)\sin\theta\, e^{-j2r}\,\mathbf{a}_\phi$ A/m in free space, find the average power passing outward through the surface $r = 10^6$, $0 < \theta < \pi/3$, and $0 < \phi < 2\pi$.

$$< \mathbf{S} >= \frac{1}{2}\text{Re}\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{15\sin^2\theta}{2\pi r^2}\mathbf{a}_r\ \text{W/m}^2$$

Then, the requested power will be

$$\Phi = \int_0^{2\pi}\int_0^{\pi/3} \frac{15\sin^2\theta}{2\pi r^2}\mathbf{a}_r \cdot \mathbf{a}_r\, r^2 \sin\theta d\theta d\phi = 15\int_0^{\pi/3} \sin^3\theta\, d\theta$$
$$= 15\left(-\frac{1}{3}\cos\theta(\sin^2\theta + 2)\right)\Big|_0^{\pi/3} = \frac{25}{8} = \underline{3.13\ \text{W}}$$

Note that the radial distance at the surface, $r = 10^6$ m, makes no difference, since the power density dimishes as $1/r^2$.

12.21. The cylindrical shell, 1 cm ¡ ρ ¡ 1.2 cm, is composed of a conducting material for which $\sigma = 10^6$ S/m. The external and internal regions are non-conducting. Let $H_\phi = 2000$ A/m at $\rho = 1.2$ cm.

a) Find **H** everywhere: Use Ampere's circuital law, which states:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho(2000) = 2\pi(1.2 \times 10^{-2})(2000) = 48\pi \text{ A} = I_{encl}$$

Then in this case

$$\mathbf{J} = \frac{I}{Area}\mathbf{a}_z = \frac{48}{(1.44 - 1.00) \times 10^{-4}}\mathbf{a}_z = 1.09 \times 10^6\,\mathbf{a}_z \quad \text{A/m}^2$$

With this result we again use Ampere's circuital law to find **H** everywhere within the shell as a function of ρ (in meters):

$$H_{\phi 1}(\rho) = \frac{1}{2\pi\rho}\int_0^{2\pi}\int_{.01}^{\rho} 1.09 \times 10^6 \;\rho\, d\rho\, d\phi = \underline{\frac{54.5}{\rho}(10^4\rho^2 - 1) \text{ A/m} \quad (.01 < \rho < .012)}$$

Outside the shell, we would have

$$H_{\phi 2}(\rho) = \frac{48\pi}{2\pi\rho} = \underline{24/\rho \quad \text{A/m} \quad (\rho > .012)}$$

Inside the shell ($\rho < .01$ m), $H_\phi = 0$ since there is no enclosed current.

b) Find **E** everywhere: We use

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{1.09 \times 10^6}{10^6}\mathbf{a}_z = \underline{1.09\,\mathbf{a}_z \quad \text{V/m}}$$

which is valid, presumeably, outside as well as inside the shell.

c) Find **S** everywhere: Use

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = 1.09\,\mathbf{a}_z \times \frac{54.5}{\rho}(10^4\rho^2 - 1)\,\mathbf{a}_\phi$$
$$= \underline{-\frac{59.4}{\rho}(10^4\rho^2 - 1)\,\mathbf{a}_\rho \quad \text{W/m}^2 \quad (.01 < \rho < .012\,\text{m})}$$

Outside the shell,

$$\mathbf{S} = 1.09\,\mathbf{a}_z \times \frac{24}{\rho}\,\mathbf{a}_\phi = \underline{-\frac{26}{\rho}\,\mathbf{a}_\rho \quad \text{W/m}^2 \quad (\rho > .012\,\text{m})}$$

12.22. The inner and outer dimensions of a copper coaxial transmission line are 2 and 7 mm, respectively. Both conductors have thicknesses much greater than δ. The dielectric is lossless and the operating frequency is 400 MHz. Calculate the resistance per meter length of the:

a) inner conductor: First

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(4 \times 10^8)(4\pi \times 10^{-7})(5.8 \times 10^7)}} = 3.3 \times 10^{-6}\text{m} = 3.3\mu\text{m}$$

Now, using (70) with a unit length, we find

$$R_{in} = \frac{1}{2\pi a \sigma \delta} = \frac{1}{2\pi(2 \times 10^{-3})(5.8 \times 10^7)(3.3 \times 10^{-6})} = \underline{0.42 \text{ ohms/m}}$$

b) outer conductor: Again, (70) applies but with a different conductor radius. Thus

$$R_{out} = \frac{a}{b} R_{in} = \frac{2}{7}(0.42) = \underline{0.12 \text{ ohms/m}}$$

c) transmission line: Since the two resistances found above are in series, the line resistance is their sum, or $R = R_{in} + R_{out} = \underline{0.54 \text{ ohms/m}}$.

12.23. A hollow tubular conductor is constructed from a type of brass having a conductivity of 1.2×10^7 S/m. The inner and outer radii are 9 mm and 10 mm respectively. Calculate the resistance per meter length at a frequency of

a) dc: In this case the current density is uniform over the entire tube cross-section. We write:

$$R(\text{dc}) = \frac{L}{\sigma A} = \frac{1}{(1.2 \times 10^7)\pi(.01^2 - .009^2)} = \underline{1.4 \times 10^{-3} \ \Omega/\text{m}}$$

b) 20 MHz: Now the skin effect will limit the effective cross-section. At 20 MHz, the skin depth is

$$\delta(20\text{MHz}) = [\pi f \mu_0 \sigma]^{-1/2} = [\pi(20 \times 10^6)(4\pi \times 10^{-7})(1.2 \times 10^7)]^{-1/2} = 3.25 \times 10^{-5} \text{ m}$$

This is much less than the outer radius of the tube. Therefore we can approximate the resistance using the formula:

$$R(20\text{MHz}) = \frac{L}{\sigma A} = \frac{1}{2\pi b \delta} = \frac{1}{(1.2 \times 10^7)(2\pi(.01))(3.25 \times 10^{-5})} = \underline{4.1 \times 10^{-2} \ \Omega/\text{m}}$$

c) 2 GHz: Using the same formula as in part b, we find the skin depth at 2 GHz to be $\delta = 3.25 \times 10^{-6}$ m. The resistance (using the other formula) is $R(2\text{GHz}) = \underline{4.1 \times 10^{-1} \ \Omega/\text{m}}$.

12.24a. Most microwave ovens operate at 2.45 GHz. Assume that $\sigma = 1.2 \times 10^6$ S/m and $\mu_r = 500$ for the stainless steel interior, and find the depth of penetration:

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(2.45 \times 10^9)(4\pi \times 10^{-7})(1.2 \times 10^6)}} = 9.28 \times 10^{-6}\text{m} = 9.28\mu\text{m}$$

b) Let $E_s = 50\angle\, 0^\circ$ V/m at the surface of the conductor, and plot a curve of the amplitude of E_s vs. the angle of E_s as the field propagates into the stainless steel: Since the conductivity is high, we use (62) to write $\alpha \doteq \beta \doteq \sqrt{\pi f \mu \sigma} = 1/\delta$. So, assuming that the direction into the conductor is z, the depth-dependent field is written as

$$E_s(z) = 50e^{-\alpha z}e^{-j\beta z} = 50e^{-z/\delta}e^{-jz/\delta} = \underbrace{50\exp(-z/9.28)}_{\text{amplitude}}\exp(-j\underbrace{z/9.28}_{\text{angle}})$$

where z is in microns. Therefore, the plot of amplitude versus angle is simply a plot of e^{-x} versus x, where $x = z/9.28$; the starting amplitude is 50 and the $1/e$ amplitude (at $z = 9.28$ μm) is 18.4.

12.25. A good conductor is planar in form and carries a uniform plane wave that has a wavelength of 0.3 mm and a velocity of 3×10^5 m/s. Assuming the conductor is non-magnetic, determine the frequency and the conductivity: First, we use

$$f = \frac{v}{\lambda} = \frac{3 \times 10^5}{3 \times 10^{-4}} = 10^9 \text{ Hz} = \underline{1 \text{ GHz}}$$

Next, for a good conductor,

$$\delta = \frac{\lambda}{2\pi} = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad \Rightarrow \quad \sigma = \frac{4\pi}{\lambda^2 f \mu} = \frac{4\pi}{(9 \times 10^{-8})(10^9)(4\pi \times 10^{-7})} = \underline{1.1 \times 10^5 \text{ S/m}}$$

12.26. The dimensions of a certain coaxial transmission line are $a = 0.8$mm and $b = 4$mm. The outer conductor thickness is 0.6mm, and all conductors have $\sigma = 1.6 \times 10^7$ S/m.

a) Find R, the resistance per unit length, at an operating frequency of 2.4 GHz: First

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(2.4 \times 10^8)(4\pi \times 10^{-7})(1.6 \times 10^7)}} = 2.57 \times 10^{-6}\text{m} = 2.57\mu\text{m}$$

Then, using (70) with a unit length, we find

$$R_{in} = \frac{1}{2\pi a \sigma \delta} = \frac{1}{2\pi(0.8 \times 10^{-3})(1.6 \times 10^7)(2.57 \times 10^{-6})} = 4.84 \text{ ohms/m}$$

The outer conductor resistance is then found from the inner through

$$R_{out} = \frac{a}{b}R_{in} = \frac{0.8}{4}(4.84) = 0.97 \text{ ohms/m}$$

The net resistance per length is then the sum, $R = R_{in} + R_{out} = \underline{5.81 \text{ ohms/m}}$.

12.26b. Use information from Secs. 6.4 and 9.10 to find C and L, the capacitance and inductance per unit length, respectively. The coax is air-filled. From those sections, we find (in free space)

$$C = \frac{2\pi\epsilon_0}{\ln(b/a)} = \frac{2\pi(8.854 \times 10^{-12})}{\ln(4/.8)} = \underline{3.46 \times 10^{-11} \text{ F/m}}$$

$$L = \frac{\mu_0}{2\pi}\ln(b/a) = \frac{4\pi \times 10^{-7}}{2\pi}\ln(4/.8) = \underline{3.22 \times 10^{-7} \text{ H/m}}$$

c) Find α and β if $\alpha + j\beta = \sqrt{j\omega C(R + j\omega L)}$: Taking real and imaginary parts of the given expression, we find

$$\alpha = \text{Re}\left\{\sqrt{j\omega C(R + j\omega L)}\right\} = \frac{\omega\sqrt{LC}}{\sqrt{2}}\left[\sqrt{1 + \left(\frac{R}{\omega L}\right)^2} - 1\right]^{1/2}$$

and

$$\beta = \text{Im}\left\{\sqrt{j\omega C(R + j\omega L)}\right\} = \frac{\omega\sqrt{LC}}{\sqrt{2}}\left[\sqrt{1 + \left(\frac{R}{\omega L}\right)^2} + 1\right]^{1/2}$$

These can be found by writing out $\alpha = \text{Re}\left\{\sqrt{j\omega C(R + j\omega L)}\right\} = (1/2)\sqrt{j\omega C(R + j\omega L)} +$ *c.c.*, where *c.c* denotes the complex conjugate. The result is squared, terms collected, and the square root taken. Now, using the values of R, C, and L found in parts a and b, we find $\alpha = \underline{3.0 \times 10^{-2} \text{ Np/m}}$ and $\beta = \underline{50.3 \text{ rad/m}}$.

12.27. The planar surface at $z = 0$ is a brass-Teflon interface. Use data available in Appendix C to evaluate the following ratios for a uniform plane wave having $\omega = 4 \times 10^{10}$ rad/s:

a) $\alpha_{\text{Tef}}/\alpha_{\text{brass}}$: From the appendix we find $\epsilon''/\epsilon' = .0003$ for Teflon, making the material a good dielectric. Also, for Teflon, $\epsilon'_r = 2.1$. For brass, we find $\sigma = 1.5 \times 10^7$ S/m, making brass a good conductor at the stated frequency. For a good dielectric (Teflon) we use the approximations:

$$\alpha \doteq \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon'}} = \left(\frac{\epsilon''}{\epsilon'}\right)\left(\frac{1}{2}\right)\omega\sqrt{\mu\epsilon'} = \frac{1}{2}\left(\frac{\epsilon''}{\epsilon'}\right)\frac{\omega}{c}\sqrt{\epsilon'_r}$$

$$\beta \doteq \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)\right] \doteq \omega\sqrt{\mu\epsilon'} = \frac{\omega}{c}\sqrt{\epsilon'_r}$$

For brass (good conductor) we have

$$\alpha \doteq \beta \doteq \sqrt{\pi f \mu \sigma_{\text{brass}}} = \sqrt{\pi\left(\frac{1}{2\pi}\right)(4 \times 10^{10})(4\pi \times 10^{-7})(1.5 \times 10^7)} = 6.14 \times 10^5 \text{ m}^{-1}$$

Now

$$\frac{\alpha_{\text{Tef}}}{\alpha_{\text{brass}}} = \frac{1/2\,(\epsilon''/\epsilon')\,(\omega/c)\sqrt{\epsilon'_r}}{\sqrt{\pi f \mu \sigma_{\text{brass}}}} = \frac{(1/2)(.0003)(4 \times 10^{10}/3 \times 10^8)\sqrt{2.1}}{6.14 \times 10^5} = \underline{4.7 \times 10^{-8}}$$

b)

$$\frac{\lambda_{\text{Tef}}}{\lambda_{\text{brass}}} = \frac{(2\pi/\beta_{\text{Tef}})}{(2\pi/\beta_{\text{brass}})} = \frac{\beta_{\text{brass}}}{\beta_{\text{Tef}}} = \frac{c\sqrt{\pi f \mu \sigma_{\text{brass}}}}{\omega\sqrt{\epsilon'_{r\,\text{Tef}}}} = \frac{(3 \times 10^8)(6.14 \times 10^5)}{(4 \times 10^{10})\sqrt{2.1}} = \underline{3.2 \times 10^3}$$

12.27. (continued)

c)

$$\frac{v_{\text{Tef}}}{v_{\text{brass}}} = \frac{(\omega/\beta_{\text{Tef}})}{(\omega/\beta_{\text{brass}})} = \frac{\beta_{\text{brass}}}{\beta_{\text{Tef}}} = \underline{3.2 \times 10^3} \quad \text{as before}$$

12.28. A uniform plane wave in free space has electric field given by $\mathbf{E}_s = 10e^{-j\beta x}\mathbf{a}_z + 15e^{-j\beta x}\mathbf{a}_y$ V/m.

a) Describe the wave polarization: Since the two components have a fixed phase difference (in this case zero) with respect to time and position, the wave has linear polarization, with the field vector in the yz plane at angle $\phi = \tan^{-1}(10/15) = 33.7°$ to the y axis.

b) Find $\mathbf{H}_s$: With propagation in forward x, we would have

$$\mathbf{H}_s = \frac{-10}{377}e^{-j\beta x}\mathbf{a}_y + \frac{15}{377}e^{-j\beta x}\mathbf{a}_z \text{ A/m} = \underline{-26.5e^{-j\beta x}\mathbf{a}_y + 39.8e^{-j\beta x}\mathbf{a}_z \text{ mA/m}}$$

c) determine the average power density in the wave in W/m^2: Use

$$\mathbf{P}_{avg} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\left[\frac{(10)^2}{377}\mathbf{a}_x + \frac{(15)^2}{377}\mathbf{a}_x\right] = 0.43\mathbf{a}_x \text{ W/m}^2 \text{ or } P_{avg} = \underline{0.43 \text{ W/m}^2}$$

12.29. Consider a left-circularly polarized wave in free space that propagates in the forward z direction. The electric field is given by the appropriate form of Eq. (100).

a) Determine the magnetic field phasor, $\mathbf{H}_s$:
We begin, using (100), with $\mathbf{E}_s = E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z}$. We find the two components of $\mathbf{H}_s$ separately, using the two components of $\mathbf{E}_s$. Specifically, the x component of $\mathbf{E}_s$ is associated with a y component of $\mathbf{H}_s$, and the y component of $\mathbf{E}_s$ is associated with a negative x component of $\mathbf{H}_s$. The result is

$$\mathbf{H}_s = \underline{\frac{E_0}{\eta_0}\left(\mathbf{a}_y - j\mathbf{a}_x\right)e^{-j\beta z}}$$

b) Determine an expression for the average power density in the wave in W/m^2 by direct application of Eq. (77): We have

$$\mathbf{P}_{z,avg} = \frac{1}{2}Re(\mathbf{E}_s \times \mathbf{H}_s^*) = \frac{1}{2}Re\left(E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z} \times \frac{E_0}{\eta_0}(\mathbf{a}_y - j\mathbf{a}_x)e^{+j\beta z}\right)$$
$$= \underline{\frac{E_0^2}{\eta_0}\mathbf{a}_z \text{ W/m}^2} \quad \text{(assuming } E_0 \text{ is real)}$$

12.30. The electric field of a uniform plane wave in free space is given by $\mathbf{E}_s = 10(\mathbf{a}_z + j\mathbf{a}_x)e^{-j50y}$. Determine:

a) f: From the given field, we identify $\beta = 50 = \omega/c$ (in free space), so that $f = \omega/2\pi = 50c/2\pi = \underline{2.39 \text{ GHz}}$.

b) $\mathbf{H}_s$: Each of the two components of $\mathbf{E}_s$ must pair with a magnetic field vector, such that the cross product of electric with magnetic field gives a vector in the positive y direction. The overall magnitude is the electric field magnitude divided by the free space intrinsic impedance. Thus

$$\mathbf{H}_s = \frac{10}{377}\left(\mathbf{a}_x - j\mathbf{a}_z\right)e^{-j50y}$$

c) $$< \mathbf{S} >= \frac{1}{2}\mathcal{R}e\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{50}{377}\left[(\mathbf{a}_z \times \mathbf{a}_x) - (\mathbf{a}_x \times \mathbf{a}_z)\right] = \frac{100}{377}\mathbf{a}_y = \underline{0.27\,\mathbf{a}_y \text{ W/m}^2}$$

d) Describe the polarization of the wave: This can be seen by writing the electric field in real instantaneous form, and then evaluating the result at $y = 0$:

$$\mathcal{E}(0,t) = 10\left[\cos(\omega t)\,\mathbf{a}_z - \sin(\omega t)\,\mathbf{a}_x\right]$$

At $t = 0$, the field is entirely along z, and then acquires an increasing negative x component as t increases. The field therefore rotates clockwise in the $y = 0$ plane when looking back toward the plane from positive y. Since the wave propagates in the positive y direction and has equal x and z amplitudes, we identify the polarization as left circular.

12.31. A linearly-polarized uniform plane wave, propagating in the forward z direction, is input to a lossless *anisotropic* material, in which the dielectric constant encountered by waves polarized along y (ϵ_{ry}) differs from that seen by waves polarized along x (ϵ_{rx}). Suppose $\epsilon_{rx} = 2.15$, $\epsilon_{ry} = 2.10$, and the wave electric field at input is polarized at 45° to the positive x and y axes. *Assume free space wavelength* λ.

a) Determine the shortest length of the material such that the wave as it emerges from the output end is circularly polarized: With the input field at 45°, the x and y components are of equal magnitude, and circular polarization will result if the phase difference between the components is $\pi/2$. Our requirement over length L is thus $\beta_x L - \beta_y L = \pi/2$, or

$$L = \frac{\pi}{2(\beta_x - \beta_y)} = \frac{\pi c}{2\omega(\sqrt{\epsilon_{rx}} - \sqrt{\epsilon_{ry}})}$$

With the given values, we find,

$$L = \frac{(58.3)\pi c}{2\omega} = 58.3\,\frac{\lambda}{4} = \underline{14.6\,\lambda}$$

b) Will the output wave be right- or left-circularly-polarized? With the dielectric constant greater for x-polarized waves, the x component will lag the y component in time at the output. The field can thus be written as $\mathbf{E} = E_0(\mathbf{a}_y - j\mathbf{a}_x)$, which is left circular polarization.

12.32. Suppose that the length of the medium of Problem 12.31 is made to be *twice* that as determined in the problem. Describe the polarization of the output wave in this case: With the length doubled, a phase shift of π radians develops between the two components. At the input, we can write the field as $\mathbf{E}_s(0) = E_0(\mathbf{a}_x + \mathbf{a}_y)$. After propagating through length L, we would have,

$$\mathbf{E}_s(L) = E_0[e^{-j\beta_x L}\mathbf{a}_x + e^{-j\beta_y L}\mathbf{a}_y] = E_0 e^{-j\beta_x L}[\mathbf{a}_x + e^{-j(\beta_y-\beta_x)L}\mathbf{a}_y]$$

where $(\beta_y - \beta_x)L = -\pi$ (since $\beta_x > \beta_y$), and so $\mathbf{E}_s(L) = E_0 e^{-j\beta_x L}[\mathbf{a}_x - \mathbf{a}_y]$. With the reversal of the y component, the wave polarization is rotated by 90°, but is still linear polarization.

12.33. Given a wave for which $\mathbf{E}_s = 15e^{-j\beta z}\mathbf{a}_x + 18e^{-j\beta z}e^{j\phi}\mathbf{a}_y$ V/m, propagating in a medium characterized by complex intrinsic impedance, η.

a) Find $\mathbf{H}_s$: With the wave propagating in the forward z direction, we find:

$$\mathbf{H}_s = \frac{1}{\eta}\left[-18e^{j\phi}\mathbf{a}_x + 15\mathbf{a}_y\right]e^{-j\beta z} \text{ A/m}$$

b) Determine the average power density in W/m^2: We find

$$P_{z,avg} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\text{Re}\left\{\frac{(15)^2}{\eta^*} + \frac{(18)^2}{\eta^*}\right\} = 275\,\text{Re}\left\{\frac{1}{\eta^*}\right\} \text{ W/m}^2$$

12.34. Given the general elliptically-polarized wave as per Eq. (93):

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{j\phi}\mathbf{a}_y]e^{-j\beta z}$$

a) Show, using methods similar to those of Example 12.7, that a linearly polarized wave results when superimposing the given field and a phase-shifted field of the form:

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{-j\phi}\mathbf{a}_y]e^{-j\beta z}e^{j\delta}$$

where δ is a constant: Adding the two fields gives

$$\begin{aligned}\mathbf{E}_{s,tot} &= \left[E_{x0}\left(1 + e^{j\delta}\right)\mathbf{a}_x + E_{y0}\left(e^{j\phi} + e^{-j\phi}e^{j\delta}\right)\mathbf{a}_y\right]e^{-j\beta z} \\ &= \left[E_{x0}e^{j\delta/2}\underbrace{\left(e^{-j\delta/2} + e^{j\delta/2}\right)}_{2\cos(\delta/2)}\mathbf{a}_x + E_{y0}e^{j\delta/2}\underbrace{\left(e^{-j\delta/2}e^{j\phi} + e^{-j\phi}e^{j\delta/2}\right)}_{2\cos(\phi-\delta/2)}\mathbf{a}_y\right]e^{-j\beta z}\end{aligned}$$

This simplifies to $\mathbf{E}_{s,tot} = 2\left[E_{x0}\cos(\delta/2)\mathbf{a}_x + E_{y0}\cos(\phi - \delta/2)\mathbf{a}_y\right]e^{j\delta/2}e^{-j\beta z}$, which is linearly polarized.

b) Find δ in terms of ϕ such that the resultant wave is polarized along x: By inspecting the part a result, we achieve a zero y component when $2\phi - \delta = \pi$ (or odd multiples of π).